

# Vector Autoregressions: Forecasting and Reality

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**C**ONSTRUCTING FORECASTS OF THE FUTURE PATH FOR ECONOMIC SERIES SUCH AS REAL (INFLATION-ADJUSTED) GROSS DOMESTIC PRODUCT (GDP) GROWTH, INFLATION, AND UNEMPLOYMENT FORMS A LARGE PART OF APPLIED ECONOMIC ANALYSIS FOR BUSINESS AND GOVERNMENT. THERE ARE A VARIETY OF METHODS AVAILABLE FOR GENERATING ECONOMIC FORECASTS. ONE COMMON TYPE OF FORECAST IS A SO-CALLED JUDGMENT-BASED FORECAST. THIS TYPE OF FORECAST IS PREDOMINANTLY THE RESULT OF A PARTICULAR FORECASTER'S SKILL AT READING THE ECONOMIC TEA LEAVES, INTERPRETING ANECDOTAL EVIDENCE, AND HIS OR HER EXPERIENCE AT OBSERVING EMPIRICAL REGULARITIES IN THE ECONOMY.

One difficulty with judgmental forecasts, however, is that it is hard, if not impossible, for an outside observer to trace the source of systematic forecast errors because there is no formal model of how the data were used. Moreover, the accuracy of judgmental forecasts can be evaluated only after a track record is established. In addition, it would not be surprising, given the element of subjectivity in such forecasts, to find that changes in the personnel of the forecasting staff substantially affected the accuracy of judgmental forecasts.

Model-based forecasts provide an alternative. They are easier to replicate and validate by independent researchers than forecasts based on expert opinion alone, and the forecaster can formally investigate the source of systematic errors in the forecasts. An important aspect of a forecast from a model that quantifies future uncertainty is that it allows the forecaster to give not only an estimate of the most likely future outcome but also a

probabilistic assessment of a range of alternative outcomes. In this context, to say that GDP growth next year is predicted to be 2 percent conveys somewhat less information about the future than does saying that GDP growth next year is most likely to be 2 percent and the probability of negative growth is less than 10 percent. Another advantage to employing model-based forecasts is that the accuracy of the point forecasts from the model can be statistically evaluated prior to using the forecasts. In other words, a forecast model's performance can be established before it is used by a decision maker. The distinction between judgmental and model-based forecasts cannot be pushed too far, however, because successful model specifications also depend heavily on the skill and ingenuity of particular individuals. No model can be left on automatic pilot for long.<sup>1</sup>

This article describes a particular model-based forecasting approach. The model studied is a vector autore-

gression (VAR) of six U.S. macroeconomic variables. Although this model is small and highly aggregated, it provides a convenient framework for illustrating several practical forecasting issues. In emphasizing the practical problems of forecasting economic data using a statistical model, the research draws on experience in using such a model at the Federal Reserve Bank of Atlanta. The focus on a simple model is intended to provide potential users with a road map of how one might implement a VAR-based forecasting model more generally.

The article first describes some practical problems in using a VAR model for forecasting purposes. The discussion focuses on a VAR model fitted to monthly data having staggered release dates that uses a distributed monthly estimate of quarterly GDP data. The article then evaluates the relative forecast performance of various alternative specifications of the VAR and offers conclusions based on the study's findings.

### Developing a VAR Model for Forecasting

The starting point for any forecasting project should be the question, What is the objective of the forecasting exercise? This question inevitably raises additional questions, such as who will be using the model and for what purpose the forecast will be used. A forecaster will design a model to fit the demands of his or her client. In essence, the end user of the forecast typically determines the variables to be incorporated into the model.

**Who Are the Clients and What Data Should Be Forecast?** The main client for models designed at the Federal Reserve Bank of Atlanta is the bank president, and his needs determine the model's design. The president serves on the Federal Open Market Committee (FOMC), the voting body of the Fed that determines monetary policy. To support his responsibilities in contributing to policy decisions, a helpful model will be designed to forecast the main economic aggregates of concern to the FOMC, such as measures of inflation, of the employment situation, and of real output.<sup>2</sup> As in Zha (1998), the VAR model described in this article includes real GDP as a measure of real output, the consumer price index (CPI) for urban households as a measure of inflation, and the civilian unemployment rate (UR) as a measure of unemployment. In addition, since it is con-

structed to help guide monetary policy, two monetary variables are included—the effective federal funds rate (FF), which is a series that the FOMC influences directly, and the M2 money stock, a series that the FOMC influences somewhat less directly. Finally, to allow a role for commodity prices in predicting future inflation, a commodity price series is also included in the variables list.<sup>3</sup>

**Handling Mixed-Frequency Data.** The VAR model is specified for data at a monthly frequency. But one of the variables in the model, real GDP, is measured only quarterly. Incorporating data of different frequencies into a single-frequency model is a vexing problem. One simple approach is to take all the higher-frequency data and convert them to the frequency of the lowest-frequency data, in this case real GDP. The estimated model would then be a quarterly model in which the other variables would typically be averages of the daily, weekly, or monthly observations. This method is a standard approach in forecasting models involving GDP data.

There is some evidence, however, that incorporating timely, monthly data can help forecast quarterly data (Miller and Chin 1996; Ingenito and Trehan 1996; Tallman and Peterson 1998).<sup>4</sup> The ability to incorporate high-frequency data into forecasts is perhaps one of the main justifications for using judgmental forecasts. However, measuring the marginal contribution of such procedures in judgmentally adjusted forecasts is difficult because the impacts of using high-frequency data on the forecasts cannot be clearly traced.

The technologies that are available for exploiting monthly information for model-based forecasts of real economic aggregates (like real GDP) take a variety of forms. The approach taken here is to use the distribution technique of Chow and Lin (1971) in constructing a monthly real GDP series. The procedure uses monthly

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1. Reifschneider, Stockton, and Wilcox (1997) provide a thorough discussion of the interaction between judgmental and model-based forecasting as practiced at the Board of Governors of the Federal Reserve System.
2. For a monetary policymaker, the Humphrey-Hawkins legislation suggests that the objectives of the Fed (as legislated by Congress) should be consistent with the federal government's goals of full employment and low inflation.
3. Exact definitions of the series used are contained in Appendix 1.
4. Ingenito and Trehan estimate a monthly model of the U.S. economy. Miller and Chin combine forecasts of, say, one month of a monthly series with two actual monthly observations within a quarter, to form preliminary estimates of the quarterly observations. In essence, there are alternative ways to extract information from monthly data and incorporate it into quarterly forecasts. Tallman and Peterson show that incorporating timely monthly information improves the forecast accuracy of a quarterly model for Australian GDP.

data on variables related to GDP (specifically, industrial production, nonagricultural payroll employment, and personal consumption expenditures) to estimate the coefficients of a regression equation for GDP at a quarterly frequency.<sup>5</sup> The regression is then used to construct estimates of monthly real GDP in a way that ensures that the quarterly average of the resulting monthly GDP estimates equals the corresponding quarterly observation of GDP.

Three new values for the monthly GDP index can be constructed with the release of GDP data for a new quarter. GDP and the data on the monthly indicator series are revised on a reasonably regular schedule (see

Appendix 1). Hence, the whole index could be reestimated every month as the existing quarterly GDP data and data on the monthly series are revised and when a new GDP observation becomes available. A description of how to implement this procedure is presented in Appendix 2.

#### Specification and Estimation of the VAR.

The idea underlying forecasting with a vector

autoregression model is first to summarize the dynamic correlation patterns among observed data series and then use this summary to predict likely future values for each series. Mathematically, a VAR expresses the current value of each of  $m$  series as a weighted average of the recent past of all the series plus a term that contains all the other influences on the current values. A VAR can be written compactly as

$$y_t = \nu + B_1 y_{t-1} + \dots + B_p y_{t-p} + u_t,$$

where  $y_t$  denotes the  $m \times 1$  vector of variables included in the VAR for month  $t$  and where all but the interest rate and unemployment are expressed in natural logarithms.<sup>6</sup> Notice that the  $m \times 1$  error vector  $u_t$  measures the extent to which  $y_t$  cannot be determined exactly as a linear combination of the past values of  $y$  with weights given by the constant coefficients  $\nu$  and  $B_l$ ,  $l = 1, \dots, p$ . Uncertainty about the value of  $u_t$  arises because the numbers of lagged observations of  $y$  to be included in the VAR,  $p$ , along with the values of the coefficients are unknown and hence will have to be estimated from the available data. The uncertainty about  $u_t$  is made operational by assuming that  $u_t$  is a random vector having a zero mean, the error covariance matrix  $\Sigma$  is positive-definite, and  $u_t$  is uncorrelated with lagged values of  $y_t$ .

It is not uncommon to find that VAR models freely fitted to data of the type used here have many estimated coefficients whose standard errors are large. Perhaps they are large because the coefficients are actually zero as indicated. Alternatively, the data might not be rich enough to provide sufficiently precise estimates of nonzero coefficients. If the parameters are too imprecise, then the situation is serious because it has been observed that large estimation uncertainty can lead to poor forecasts.<sup>7</sup> Getting imprecise parameter estimates in a VAR is likely to be a common practical problem because the number of parameters is often quite large relative to the available number of observations. For example, in the next section of the article the VAR models are specified with  $p$  as large as 13. With six variables in the VAR, a total of seventy-nine coefficients would therefore be estimated in each equation in the VAR. Various solutions to the problem of “overfitting” VAR models have been proposed in the forecasting literature, and these all amount to putting prior constraints on the values of the model’s coefficients so as to require less information from the data when determining the coefficient values. These prior restrictions act as nondata information regarding the coefficient values.

One approach to reducing the coefficient uncertainty is to set some coefficients to zero or other pre-assigned values. These values may or may not have been determined on the basis of prior fitting of models to the data. For example, one might prespecify a maximum lag order  $p_{\max}$  for the VAR ( $p_{\max} = 13$  in the empirical analysis) and select the  $p \leq p_{\max}$  that minimizes a specific criterion. This criterion discounts the increase in measured in-sample fit that occurs simply because one is fitting more coefficients to a fixed set of observations. Another way that coefficient restrictions are used in a VAR forecasting model is by predifferencing data series that appear to exhibit trends or quite persistent local levels over time, prior to fitting the VAR. This approach would be mathematically the same as imposing exact restrictions on the coefficients of a VAR in the levels of the data.

An alternative approach is to impose inexact prior restrictions. For example, rather than setting all lags greater than  $p$  to zero, the VAR could be estimated in a way that gives more weight to nonzero coefficients on recent observations relative to those on more distant lags but without necessarily setting the coefficients on the more distant lags to zero. Similarly, the degree of uncertainty about the constraints implied by predifferencing the data could be incorporated explicitly into the coefficient estimation strategy.

The methods presented in Litterman (1980, 1986), Doan, Litterman, and Sims (1984), Sims (1992), Kadiyala and Karlsson (1993, 1997), and Sims and Zha (1998) are all essentially ways of imposing inexact prior

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restrictions on the coefficients of a VAR. How closely the prior restrictions are imposed is usually determined by examining historical out-of-sample forecast performance across various degrees of tightness on the prior restrictions. The specification can be made to resemble an exact restriction if the resulting improvement in forecast performance warrants such a specification, but it has been found that the best performance usually arises by not imposing the restrictions exactly. The implementation of various types of inexact prior restriction is described in some detail in Box 1.

#### **Handling the Staggered Timing of Data Releases.**

Another data-related problem is that new data are released at irregular intervals. For example, an average interest rate for a month is available at the end of the month being measured. On the other hand, a money stock estimate for a month is not available until the middle of the following month. Moreover, although the distribution of the real GDP series puts the model data on a monthly frequency, the new GDP observations can be estimated only toward the end of the month after the quarter being measured. In order to exploit monthly data that are available on some but not all series in the VAR, the so-called conditional forecasting technique, as described in Doan, Litterman, and Sims (1984) and Litterman (1984), is used. In this framework, at the end of a particular month, say, the value for all data series that are not yet available for that month are forecast “conditional” upon all the variables for which observations are available for the current month. The procedure involves first estimating the VAR model using a sample that contains complete observations on all the variables in the model. At the end of January, then, the VAR would be estimated with data up to the previous December. Then a forecast of all the variables from the VAR for January is made as if no additional data were available. However, the forecasts for the federal funds rate and commodity prices must be exact because their January values are at hand, and this information should allow deducing more accurate forecasts of the other series whose January values are not known. The size of the improvement to the forecast for January will depend on the extent to which knowing the values of January’s federal funds rate and commodity prices is useful for predicting the other series’ values for that month. This idea can be readily extended to more complicated situations,

as occurs at the end of March when the values for the federal funds rate and commodity prices for January, February, and March and values of M2, CPI, and unemployment for January and February are available but there is no value for first-quarter GDP.<sup>8</sup> A simple example that illustrates the implementation of the foregoing conditional forecasting procedure is presented in Box 2.

**Measuring Forecast Accuracy.** Evaluating the accuracy of forecasts is a form of accountability in the sense that a client would presumably give more weight to a forecasting scheme that can generate relatively accurate forecasts. In addition, forecast evaluation is relevant to the forecaster when deciding on a model specification for subsequent use. The preferences or loss function of the forecast user is key to the selection of the accuracy criterion. In most forecast evaluations the accuracy measures are some form of average error, typically root mean squared error (RMSE) or mean absolute error (MAE), but many other possibilities are available. For example, the proportion of times the direction of a change in a variable is correctly forecast may be relevant for evaluating forecasts if, for example, capturing turning points were viewed as of primary importance. The results reported below use the RMSE as the accuracy criterion, but it is acknowledged that using other forecast accuracy criteria may yield different model rankings.

In justifying the final specification to a client, most forecasters would present some evidence regarding the accuracy and reliability of the model over some historical period. This evidence is likely to be a by-product of the model selection process given that a forecaster will probably have spent considerable time conducting historical out-of-sample forecasting experiments in order to tune the model specification. In the empirical application presented below, the period from 1986 to 1997 is used to examine the forecast performance of the various VAR specifications.

**The idea underlying forecasting with a vector autoregression model is first to summarize the dynamic correlation patterns among observed data series and then use this summary to predict likely future values for each series.**

5. *These monthly data are the same as those used in constructing the Conference Board’s coincident index of economic activity except that this model uses consumption expenditure instead of disposable income and does not use the monthly retail and trade sales series because it has a two-month release lag.*
6. *The selection of the variables included in the model implies a strong exclusion restriction on all other variables that could have been but were not included.*
7. *On this point see Wallis (1989) and Fair and Shiller (1990).*
8. *An algorithm for implementing this procedure is available in the RATS software package. The authors have a more general version programmed using the GAUSS language that allows a wider range of conditioning experiments.*

# Inexact Prior Restrictions in VAR Forecasting Models

As described in the text, a VAR model for the  $m \times 1$  vector of observations  $y_t$  has the form

$$y_t = \nu + B_1 y_{t-1} + \dots + B_p y_{t-p} + u_t, \quad (B1)$$

where the coefficients  $\nu$  and  $B_l$ ,  $l = 1, \dots, p$ , and the covariance of  $u_t$ ,  $\Sigma$ , are to be estimated once a value for  $p$  is specified. A major problem with using a VAR such as equation (B1) for forecasting based on unrestricted OLS estimates of the coefficients when  $p$  is moderately large is that the coefficient values are often not very well determined in a finite set of data. Litterman (1980, 1986) discusses this problem in the context of economic series that exhibit trends or persistent local levels and suggests an alternative, Bayesian, method for estimating the coefficients in these cases. The idea is to treat the coefficients as random quantities around given mean values, with the tightness of the distributions about these prior means determined via a set of hyperparameters. The OLS coefficient estimator is then modified to incorporate the inexact prior information contained in these distributions. The main technical issues involve specifying the form of the prior distributions and determining the form of the estimators.

Litterman's method is often referred to as the Minnesota prior because of its origins at the University of Minnesota and the Federal Reserve Bank of Minneapolis. It is usually implemented as follows.

The prior for the individual elements of each lag coefficient matrix  $B_l$  is that they are each independent, normally distributed random variables with the mean of the coefficient matrix on the first lag,  $B_1$ , equal to an identity matrix and the mean of the elements of  $B_l$ ,  $l > 1$ , equal to zero. Notice that if these restrictions were exact then each variable would be a random walk, possibly with nonzero drift. While the random walk prior might be considered a reasonable specification, there is no need to impose it exactly on the VAR. In particular, the standard deviation of the  $ij$ -th element of the  $l$ -th lag coefficient matrix  $B_l$  can be nonzero, with these being often specified as  $\lambda_i/l^{\lambda_3}$ , if  $i = j$ , and  $\sigma_i \lambda_1 \lambda_2 / \sigma_j l^{\lambda_3}$ , if  $i \neq j$  (see, for example, Sims 1992).

The parameter  $\lambda_1$  is the prior standard deviation of the  $ii$ -th element of  $B_1$ , reflecting how closely the random walk approximation is to be imposed. Lowering  $\lambda_1$  toward zero has the effect of shrinking the diagonal elements of  $B_1$  toward one and all other coefficients to zero.

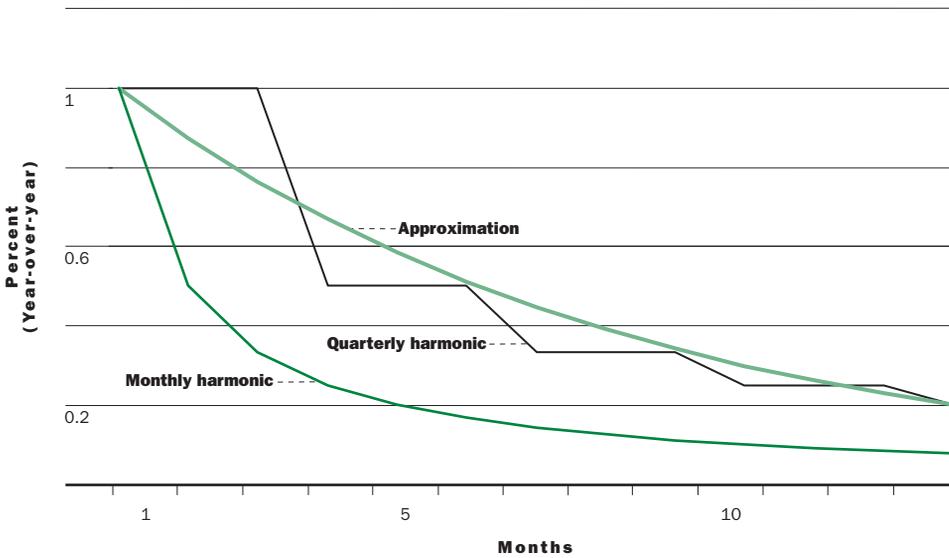
It is reasonable to suppose that most of the variation in each of the dependent variables in the VAR is accounted for by its own lags. Thus, in each row of  $B_l$ , coefficients on lags of other variables can be assigned smaller variance in relative terms by choosing a value for  $\lambda_2$  such that  $0 < \lambda_2 \leq 1$ . Decreasing  $\lambda_2$  toward zero has the effect of shrinking the off-diagonal elements of  $B_l$  toward zero. Setting  $\lambda_2$  to unity means that no distinction is made between the lags of the dependent variable and the lags of other variables.

The ratio  $(\sigma_i/\sigma_j)$  is included in the prior standard deviations to account for the differences in the units of measurement of different variables. If the variability of  $y_{i,t}$  is much lower than that of  $y_{j,t}$ , then the coefficient on  $y_{j,t-1}$  in the  $i$ -th equation is shrunk toward zero. In practice, the  $\sigma_i$  are usually set equal to the residual standard error from an OLS regression of each dependent variable on  $p$  lagged values.

The parameter  $\lambda_3 > 0$  is used to determine the extent to which coefficients on lags beyond the first one are likely to be different from zero. As  $\lambda_3$  increases, the coefficients on high-order lags are being shrunk toward zero more tightly. If  $\lambda_3$  is set to one, the rate of decay in the weight is harmonic. For a VAR fitted to monthly data one could choose to use a decay rate that approximates a harmonic decay pattern at a quarterly frequency. In particular, in a monthly VAR with  $p = 13$  lags, rather than using  $l^{-1}$ , the decay could be specified as  $\exp(cl - c)$  where  $c = -0.13412$ . This approximation is depicted in Chart A, which shows the harmonic decay pattern at a quarterly frequency ( $1, 1/2, 1/3, 1/4, 1/5$ ). The approximately equivalent monthly decay is based on  $p = 13$  lags. Notice that this specification ensures that for  $l = 13$  the decay rate is exactly  $1/5$ . By contrast, assuming a harmonic decay at the monthly frequency results in a much faster rate of decay, as the chart shows. In fact, under the monthly harmonic decay scheme the weight given to the coefficients on observations from five months ago would be the same as that given to the coefficients attached to observations from thirteen months ago under the quarterly harmonic decay approximation. The choice of the decay pattern will undoubtedly have some influence on the model's forecast performance.

As for the constant term  $\nu$ , there are numerous ways that an inexact prior restriction could be implemented. Here the prior mean of the constant in the  $i$ -th equation is taken to be zero with standard deviation  $\sigma_i \lambda_4$ . As  $\lambda_4$  decreases, the constant is shrunk toward zero.

CHART A Lag Decay Rates



To summarize, the Minnesota prior for the coefficients of the  $i$ -th equation of the VAR is that the coefficient vector  $b_i$  is normally distributed with mean  $\bar{b}_i$  and covariance  $\bar{G}_i$ . For example, if  $m = 2$  variables and there are  $p = 2$  lags of each variable in the VAR, then  $\bar{b}_1 = (0 \ 1 \ 0 \ 0 \ 0)'$  and  $\bar{b}_2 = (0 \ 0 \ 1 \ 0 \ 0)'$ . The diagonal prior covariance matrix  $\bar{G}_1$  has nonzero elements  $(\sigma_1 \lambda_4)^2$ ,  $\lambda_1^2$ ,  $(\sigma_1 \lambda_1 \lambda_2 / \sigma_2)^2$ ,  $(\lambda_1 / 2^{\lambda_3})^2$ , and  $[\sigma_1 \lambda_1 \lambda_2 / (2^{\lambda_3} \sigma_2)]^2$  while  $\bar{G}_2$  has nonzero elements  $(\sigma_2 \lambda_4)^2$ ,  $(\sigma_2 \lambda_1 \lambda_2 / \sigma_1)^2$ ,  $\lambda_2^2$ ,  $[\sigma_2 \lambda_1 \lambda_2 / (2^{\lambda_3} \sigma_1)]^2$ , and  $(\lambda_1 / 2^{\lambda_3})^2$ .

In terms of coefficient estimation, one should note that the usual OLS estimator of the coefficients of the  $i$ -th equation of the VAR model in equation (B1) has the form

$$\hat{b}_i^{OLS} = (X'X)^{-1} X'y_i, \quad i = 1, \dots, m,$$

where  $y_i$  is a  $T \times 1$  vector with  $T$ -th element  $(y_{i,T})$  and  $X$  is a  $T \times (mp + 1)$  matrix with  $T$ -th row  $(1 \ y_{1,T-1} \dots y_{m,T-1} \ y_{1,T-2} \dots y_{m,T-2} \dots y_{m,T-p})$ . In contrast, the coefficient estimator, or more formally, the mean of the posterior distribution under the Minnesota prior, is

$$\hat{b}_i^{LP} = (\bar{G}_i^{-1} + \sigma_i^{-2} X'X)^{-1} (\bar{G}_i^{-1} \bar{b}_i + \sigma_i^{-2} X'y_i)$$

(see, for example, Lutkepohl 1991).

Under a strict interpretation of the Minnesota prior, the estimator of the error covariance is a diagonal matrix with  $\sigma_i$  along the diagonal and with  $\sigma_i$  determined from the data. The coefficient estimation problem is therefore simplified because it avoids having to specify how the prior distribution  $\Sigma$  is related to the prior distribution for  $B_i$ . In practice, the diagonality restriction on  $\Sigma$  is often ignored,

and a nondiagonal estimator based on the residual sum-of-squares matrix is used instead.

Over recent years Bayesian VAR techniques have been developed that remove the assumption that the error covariance matrix is fixed and diagonal (see Kadiyala and Karlsson 1993, 1997, and Sims and Zha 1998 for a discussion). For example, one could replace the Minnesota prior with a specific form of a so-called Normal-Wishart prior. Under a Normal-Wishart prior, the prior distribution of coefficients (given  $\Sigma$ ) is Normal while the prior distribution of  $\Sigma$  is inverse Wishart (see Drèze and Richard 1983, 539–41). This feature allows the random-walk aspect of the Minnesota prior on the coefficients to be used without having to take independence across the equations of the VAR as an exact restriction.

Under the Normal-Wishart prior, the coefficient estimator (the mean of the posterior distribution) has the form

$$\hat{B} = (\bar{H}^{-1} + X'X)^{-1} (\bar{H}^{-1} \bar{B} + X'y),$$

where  $\bar{B}$  is the prior mean of the coefficient matrix  $B = (b_1 \dots b_m)$ , and  $\bar{H}$  is a diagonal, positive-definite matrix, with elements defined as in Sims and Zha (1998). The corresponding estimator of the error covariance is

$$\hat{\Sigma} = T^{-1} (y'y - \hat{B}'(X'X + \bar{H}^{-1})\hat{B} + \bar{B}'\bar{H}^{-1}\bar{B} + \bar{S}),$$

where  $\bar{S}$  is the diagonal scale matrix in the prior inverse-Wishart distribution for  $\Sigma$ . As a specific example, if  $m = 2$  and  $p = 2$ , then  $\text{vec}(\bar{B}) = (0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)'$ ;  $\bar{S}$  has  $(\sigma_j / \lambda_0)^2$  along the diagonal; and  $\bar{H}$  has diagonal elements

$(\lambda_0\lambda_4)^2$ ,  $(\lambda_0\lambda_1/\sigma_1)^2$ ,  $(\lambda_0\lambda_1/\sigma_2)^2$ ,  $[\lambda_0\lambda_1/(2^{\lambda_2}\sigma_1)]^2$ , and  $[\lambda_0\lambda_1/(2^{\lambda_2}\sigma_2)]^2$ . The parameter,  $\lambda_0$ , can be thought of as controlling the overall tightness of the prior on  $\Sigma$ .<sup>1</sup>

To see how this setup is related to the Minnesota prior, note that under a Normal-Wishart prior the covariance of coefficients has a form whereby  $\bar{H}$  is multiplied by each element of  $\bar{S}$  (a Kroneker product operation). Doing this multiplication yields an  $m(mp + 1) \times m(mp + 1)$  scale matrix whose elements are exactly the coefficient prior variances under the Minnesota prior but with  $\lambda_2 = 1$ . This latter restriction is required because the Normal-Wishart prior implies a certain symmetry across the equations of the VAR (apart from scale). In particular, it prohibits the prior from treating lags of the dependent variable differently from lags of other variables in each equation. In a sense the restriction on  $\lambda_2$  is the price of being able to relax the strong error covariance assumption of the Minnesota prior while still being able to have an estimator that is simple to implement.<sup>2</sup>

Other types of inexact prior information have been introduced as modifications of the Minnesota prior that involve priors on linear combinations of the coefficients in equation (B1). Because this modification introduces non-zero off-diagonal terms into the prior covariance for the individual coefficients, it is usually implemented by mixing a set of dummy observations into the data set rather than directly specifying the prior covariance structure. The magnitude of the weight attached to the dummy observations is used to determine the tightness of the prior restriction.

Two types of initial dummy observations are discussed here. The first is motivated by the frequent practice of specifying a VAR model of data that contain stochastic trends (unit roots) in first differences of the data. This specification corresponds to the restrictions that the sums of coefficients on the lags of the dependent variable in each equation of the VAR equal one while coefficients on lags of other variables sum to zero. Formally, when  $\sum_{i=1}^p B_i = I$ , the VAR can be written as

$$\Delta y_t = v + \Gamma_1 \Delta y_{t-1} + \dots + \Gamma_{p-1} \Delta y_{t-p+1} + u_t,$$

where  $\Gamma_i = \sum_{n=i+1}^p B_n$  and forecasts of period-to-period changes in the variables will not be influenced at all by

the current level of the series. To accommodate this possibility, Doan, Litterman, and Sims (1984) developed the so-called sum of coefficients prior for a VAR specified in the levels of the data. This prior is implemented by adding a set of  $m$  initial dummy observations to the data set. For example, if there are  $m = 2$  variables then there are two sum-of-coefficients dummy observations for each equation. For the dependent variable in the first equation, these are  $(\bar{y}_1^0 \ 0)'$  while they are  $(0 \ \bar{y}_2^0)'$  for the dependent variable in the second equation, and  $\bar{y}_i^0$  is the mean of the  $p$  presample values for variable  $y_i$ . If there are  $p = 2$  lags of each of the two variables in the VAR, then the matrix of dummy observations for the regressors in both equations has the form

$$\begin{pmatrix} 0 & \bar{y}_1^0 & 0 & \bar{y}_1^0 & 0 \\ 0 & 0 & \bar{y}_2^0 & 0 & \bar{y}_2^0 \end{pmatrix}.$$

A weight of  $\mu_5 \geq 0$  is then attached to these dummy observations, and as  $\mu_5 \rightarrow \infty$  the estimated VAR will increasingly satisfy the sum of coefficients restriction.<sup>3</sup> Notice also that as  $\mu_5 \rightarrow \infty$  the forecast growth rates will eventually converge to their sample average values.

If it is supposed that each series contains a stochastic trend, then as  $\mu_5 \rightarrow \infty$  the sum of coefficients restriction implies that there are as many stochastic trends in the VAR as there are variables in  $y_t$ . However, it might be reasonable to suppose that there are fewer than  $m$  stochastic trends in the VAR, as would be the case if there were stable long-run relations between the trending series (cointegration). Sims (1992) observed that introducing an additional dummy observation could make some allowance for this possibility. If  $m = 2$  and  $p = 2$ , the Sims initial dummy observation for the dependent variable in the  $i$ -th equation is  $(\bar{y}_i^0)$  while  $(1 \ \bar{y}_1^0 \ \bar{y}_2^0 \ \bar{y}_1^0 \ \bar{y}_2^0)$  is the vector of dummy regressor observations. A weight of  $\mu_6 \geq 0$  is then attached to these dummy observations for each equation. If the series individually contain stochastic trends, then as  $\mu_6 \rightarrow \infty$  increasingly more weight will be put on a VAR that can be written in a form in which all series share a single stochastic trend and the intercept will be close to zero.<sup>4</sup>

1. For given  $\sigma_y$ , it is only the products  $\lambda_0\lambda_1$  and  $\lambda_0\lambda_4$  that have a direct influence on the coefficient estimator.  
 2. GAUSS and MATLAB code for implementing this prior is available from the authors upon request.  
 3. It is not necessary that  $\mu_5$  be the same in each equation. See, for example, Miller and Roberds (1991).  
 4. A remaining practical question is how to select values for the hyperparameters since the quality of the forecasts will depend on these choices. In practice, values are determined based on examining historical forecast performance across a range of parameter settings.

### Summary of the VAR Forecasting Procedure.

Assuming that the forecasts are constructed within a few days of the end of a month, there is a simple pattern for forecast construction across a year. In particular, forecasts formed at the end of January, April, July, and October all have the same structure. Those formed at the end of February, May, August, and November all have the same structure, and those formed at the end of March, June, September, and December have the same structure. As a specific example of the previous discussion, it may be helpful to consider how forecasts would have been formed at the end of January, February, and March 1998.

*January.* At the end of January 1998 the commodity price index and funds rate for January would be available. The most recent quarterly GDP observation would be the “advanced estimate” for the fourth quarter of 1997. The latest available CPI, unemployment rate, M2 stock, and monthly GDP data (constructed using the Chow-Lin procedure) are all for December 1997. A VAR is then fitted to the latest vintage of monthly data through December 1997. Conditional forecasts of the unemployment rate, CPI, M2, and GDP are made for January, and with these forecasts in hand forecasts of the quarterly average unemployment rate, inflation rate, and rate of GDP growth are constructed, along with corresponding forecasts of the annual averages and growth rates.

*February.* At the end of February 1998 the commodity price index and funds rate for January and February, the CPI, unemployment rate, and M2 stock for January, and a GDP estimate for the preceding fourth quarter would be in hand. The latest quarterly GDP estimate for the fourth quarter of 1997 is the “preliminary estimate.” The Chow-Lin procedure is reapplied to generate a new monthly GDP series, and the VAR is refit using the vintage of data through December 1997, available at the end of February. Conditional forecasts of the unemployment rate, CPI, and M2 are made for February, and a conditional forecast of GDP is made for January and February. Forecasts of the quarterly and annual averages/growth rates are then constructed.

*March.* At the end of March 1998 the commodity price index and funds rate for January, February, and March, the CPI, unemployment rate, and M2 stock for January and February, and a GDP estimate for the preceding fourth quarter would be available. The latest quarterly GDP estimate for the fourth quarter of 1997 is the “final estimate.” The Chow-Lin procedure is reap-

plied to generate a new monthly GDP series based on the revised data, and the VAR is fitted using the vintage of data through to December 1997 available at the end of March. Conditional forecasts of the unemployment rate, CPI, and M2 are made for March, and a conditional forecast of GDP is made for January, February and March. Forecasts of the quarterly and annual averages/growth rates are then constructed.

The process then repeats itself. For example, at the end of April 1998 the commodity price index and funds rate for April, the CPI, unemployment rate, and M2 stock for March, and a GDP estimate for the first quarter of the current year are available, and so on.

### Empirical Application

This section reports the results of using various VAR specifications to forecast the unemployment rate, the inflation rate, and the rate of growth in GDP for the current and the next

quarter and the current and next calendar year over the period from 1986 to 1997. This comparison is not intended as a formal model evaluation but simply as a demonstration that the nature of prior restrictions on a VAR specification can have important implications for forecast performance. The alternative specifications considered are

- An unrestricted VAR specification in the levels of the data, estimated by OLS, and with  $p = 13$  imposed. This specification is denoted as the OLS model.
- A VAR specification estimated by OLS, with the sum-of-coefficients restriction imposed exactly (the data are first-differenced) and with the lag length chosen on the basis of the Akaike Information Criterion (AIC) with  $p_{\max}$  set at 13 (see Lutkepohl 1991 for a discussion of the AIC). This specification is denoted as the DOLS-AIC model.<sup>9</sup>
- A Litterman VAR as described in Box 1, with the settings for the parameters of the prior standard deviations those suggested by Litterman (1986),

**Evaluating the accuracy of forecasts is a form of accountability in the sense that a client would presumably give more weight to a forecasting scheme that can generate relatively accurate forecasts.**

9. The study also examined the forecasts from a VAR in levels, where the lag length was chosen based on the AIC (denoted as OLS-AIC). The OLS-AIC model's accuracy is better than the unrestricted OLS model but not as good as the DOLS-AIC model. Similarly, an unrestricted VAR in first differences (denoted as DOLS) performs better than the OLS and OLS-AIC specifications but not as well as the DOLS-AIC model. The fact that specifications in which the lag length is selected on the basis of a penalty function outperform those that do not suggests that down-weighting distant lags is advantageous to forecast performance.

# Conditional Forecasting with A VAR

A frequent problem in implementing a VAR model for forecasting is that observations on all the variables in the model for the current month are not available. Rather than using a complete data set, which may involve using relatively old data or postponing the forecast until all new data are available, one can make a forecast of the missing observations using the partially complete data set. If there is significant high-frequency correlation among the observed and yet-to-be-observed data, then this approach should generate better forecasts of the missing observations than if that information were ignored.

As a hypothetical example of the strategy, suppose that there is a VAR with  $m = 2$  variables and  $p = 1$  lag such that each series is a random walk and the errors have unit variance and a contemporaneous (current-period) correlation of 0.25. Thus, the VAR can be written as

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix},$$

where, as at time  $t$ , the  $e_t$  are expected to be zero with covariance equal to an identity matrix. Now suppose that for some reason, possibly because of lags in the publication of data, at the end of February 1998 there are observations on  $y_1$  for December 1997 and for January and February 1998 while the only observation on  $y_2$  is for December 1997. If  $T$  denotes December 1997, then the January and February values are determined according to

$$\begin{pmatrix} y_{1T+1} \\ y_{2T+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1T} \\ y_{2T} \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} e_{1T+1} \\ e_{2T+1} \end{pmatrix},$$

and

$$\begin{pmatrix} y_{1T+2} \\ y_{2T+2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1T} \\ y_{2T} \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} e_{1T+1} \\ e_{2T+1} \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} e_{1T+2} \\ e_{2T+2} \end{pmatrix}.$$

With only data for December 1997 in hand, the best guess about the future errors is zero, and hence the forecast of January's and February's data values for variable  $y_i$  will simply be the corresponding December data values  $y_{iT}$ . However, this forecast ignores the fact that the January and February values of  $y_1$ ,  $y_{1T+1}$  and  $y_{1T+2}$  are actually already known. The problem is how to use this extra information to refine the forecasts of  $y_{2T+1}$  and  $y_{2T+2}$ .

Suppose the difference between the December forecasts of  $y_{1T+1}$  and  $y_{1T+2}$  and their actual values were stacked in a  $2 \times 1$  vector,

$$r = \begin{bmatrix} y_{1T+1} - y_{1T} \\ y_{1T+2} - y_{1T} \end{bmatrix}.$$

Notice that the first element of  $r$  is exactly equal to  $e_{1T+1}$  in the example and the second element of  $r$  is equal to  $e_{1T+1} + e_{1T+2}$ . That is, there is only one value for  $e_{1T+1}$  that can ensure that the forecast of  $y_{1T+1}$  satisfies the constraint, and hence there is only one value for  $e_{1T+2}$  that can make the second constraint hold. Denote as  $e$  the  $4 \times 1$  vector of stacked error terms,

$$e = (e_{1T+1} \ e_{2T+1} \ e_{1T+2} \ e_{2T+2})',$$

and let  $R$  be a  $2 \times 4$  matrix,

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

where the first row of  $R$  gives the relationship between the first element of  $r$  and  $e$  while the second row gives the relationship between the second element of  $r$  and  $e$ . The result is a system of equations of the form  $r = Re$  to solve for  $e$ , and in general there will be an infinity of possible solutions.<sup>1</sup>

However, Doan, Litterman, and Sims (1984) showed that a unique vector of forecast errors that both satisfy the constraints and minimize the sum of squared errors  $e'e$  is given by  $\hat{e} = R'(RR')^{-1}r$ . In a least-squares sense this is the most likely set of values for the forecast errors. In the example this solution yields

$$\hat{e} = (\hat{e}_{1T+1} \ 0 \ \hat{e}_{1T+2} \ 0)',$$

where  $\hat{e}_{1T+1} = y_{1T+1} - y_{1T}$  and  $\hat{e}_{1T+2} = y_{1T+2} - y_{1T} - \hat{e}_{1T+1}$ . Finally, substituting the elements of  $\hat{e}$  back into the VAR yields modified, or conditional, forecasts. In particular, instead of using  $y_{1T}$  as the forecast of  $y_{1T+j}$  for  $j = 1, 2$ , their actual known values would be used, and rather than using  $y_{2T}$  as the forecast of  $y_{2T+j}$ ,  $y_{2T} + 0.5\hat{e}_{1T+1}$  and  $y_{2T} + 0.5(\hat{e}_{1T+1} + \hat{e}_{1T+2})$  would be used, respectively.

Although designed to handle incomplete data matrices in a VAR model, this procedure can also be extended to allow for conditioning on assumed future paths for variables in the model by simply treating these as known data. For example, in September 1998 one might wish to condition on GDP growth to be an annualized 2 percent per quarter for the current and the next quarter. Doing so would

introduce two constraints (one for each quarter) into the VAR to be spread across six months.<sup>2</sup>

Waggoner and Zha (1998) observe that, although  $\hat{\epsilon}$  are the most likely outcomes for the forecast errors, one can also allow for the fact that over the constraint period (January and February in the example) the values of the nonconstrained errors will not necessarily be zero. In particular, the stacked forecast errors will be randomly dis-

tributed with mean  $\hat{\epsilon}$  and a singular covariance matrix given by  $I - R'(RR')^{-1}R$ . This result can be used to generate error bands for the conditional forecasts by repeatedly simulating observations from this distribution. The reader is referred to Waggoner and Zha for additional details relating to conditional forecasting in this context and for methods for accounting for uncertainty about the true values of the model's coefficients in these simulations.

1. In general,  $R$  is of dimension  $n \times (mk)$ , where  $n$  is the number of constraints to be satisfied and  $k$  is the maximum number of months the constraints are imposed.
2. This strategy assumes that one does not want to incorporate additional information as to the underlying source of the 2 percent GDP growth. See Litterman (1984) and Waggoner and Zha (1998) for a discussion on this point.

that is,  $\lambda_1 = 0.2$ ,  $\lambda_2 = 0.2$ , and  $\lambda_3 = 1$ , with  $\lambda_4$  set as 0.3.

- A modified Litterman VAR that uses the base Litterman settings but also incorporates sum-of-coefficients and cointegration dummy observations described in Box 1, with weights  $\mu_5 = 5$  and  $\mu_6 = 5$ , respectively.
- The VAR specification used in Waggoner and Zha (1998) and Zha (1998) and described in Box 1. Following Waggoner and Zha the prior standard deviation parameter values are set at  $\lambda_0 = 0.6$ ,  $\lambda_1 = 0.1$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = 1$ ,  $\lambda_4 = 0.1$ ,  $\mu_5 = 5$  and  $\mu_6 = 5$ . This model is denoted as the ZVAR model.
- A partial ZVAR specification that uses the base parameter settings of the ZVAR model but shuts off the dummy observations by setting  $\mu_5 = 0$  and  $\mu_6 = 0$ .

Each non-OLS specification uses the monthly approximation to a quarterly harmonic lag decay pattern as discussed in Box 1.

Robertson and Tallman (1998) describe how a true real-time forecast experiment would involve using exactly the data series available at the time the forecasts were made, together with a model specification and coefficients determined using these data. The results reported below are obtained by using the real-time vintage of historical data to construct the forecasts and a recent vintage of data (as of July 1998) to evaluate the forecasts. However, because the parameter setting for the ZVAR specification was chosen on the basis of out-of-sample forecast performance, the study does not accurately replicate real-time forecasts. In particular, a forecaster in 1986 would not have been able to use the post-1986 forecast performance to guide the model specification.

For each VAR specification a maximum of  $p = 13$  lagged observations of the six variables in  $y_t$  are includ-

ed in each equation. The VAR is first fit to data for the period from February 1960 to December 1985, with the thirteen presample values being those for January 1959 to January 1960. The VAR is reestimated (and lag length reselected in the case of the DOLS-AIC model) every three months through December 1997, and in doing so the coefficient estimates can vary in response to new data and revisions to existing data. As described in the previous section, each time the model is reestimated, (conditional) forecasts of the unemployment rate, inflation, and GDP growth are generated for the current and the two subsequent quarters, as well as forecasts for the current and each of the two subsequent calendar years. Pooling the forecast errors for each period (quarter or year) yields a set of 144 current-quarter forecasts, 141 next-quarter forecasts, 138 subsequent-quarter forecasts, 144 current-year forecasts, 132 forecasts for the next calendar year, and 120 forecasts for the calendar year after next. Each model's forecast accuracy is evaluated on the basis of the RMSE statistic.

**Forecast Results.** The RMSE results are reported in Table 1. The numbers in parentheses give the ratio of the RMSE of the associated model to the RMSE of the ZVAR forecasts at each horizon. A value greater than one for this ratio means that the RMSE of the given model is larger than for the ZVAR forecasts, indicating that those forecasts are less accurate. The modified Litterman model generally produces the smallest RMSE values across variables and forecast horizons. However, the ZVAR model is only slightly less accurate overall. The DOLS-AIC model also performs very well for the inflation and GDP growth forecasts but generates relatively poor unemployment forecasts. The basic Litterman and partial ZVAR models are ranked next in terms of accuracy, and the unrestricted OLS model is clearly

**TABLE 1 RMSE of VAR Forecasts 1986–97<sup>a</sup>**

	Current Quarter	First Quarter	Second Quarter	Current Year	First Year	Second Year
Unemployment						
OLS	0.190 (1.25)	0.367 (1.30)	0.546 (1.33)	0.261 (1.54)	0.920 (1.42)	1.435 (1.46)
DOLS-AIC	0.161 (1.06)	0.324 (1.15)	0.519 (1.26)	0.202 (1.20)	0.879 (1.36)	1.377 (1.40)
Litterman	0.160 (1.05)	0.306 (1.10)	0.449 (1.09)	0.220 (1.30)	0.740 (1.14)	1.089 (1.10)
Modified Litterman	0.151 (0.99)	0.277 (0.98)	0.398 (0.97)	0.166 (0.98)	0.619 (0.96)	0.940 (0.95)
ZVAR	0.152	0.282	0.411	0.169	0.647	0.986
Partial ZVAR	0.159 (1.05)	0.302 (1.07)	0.439 (1.05)	0.216 (1.28)	0.718 (1.10)	1.069 (1.08)
CPI Inflation						
OLS	1.216 (1.30)	2.105 (1.38)	2.207 (1.39)	0.552 (1.34)	1.719 (1.56)	2.560 (2.32)
DOLS-AIC	1.076 (1.15)	1.701 (1.12)	1.541 (0.97)	0.444 (1.08)	1.074 (0.98)	1.106 (1.00)
Litterman	0.973 (1.04)	1.737 (1.14)	1.787 (1.12)	0.472 (1.15)	1.227 (1.12)	1.451 (1.32)
Modified Litterman	0.920 (0.99)	1.523 (1.00)	1.616 (1.02)	0.400 (0.97)	1.106 (1.00)	1.133 (1.03)
ZVAR	0.933	1.524	1.590	0.411	1.099	1.102
Partial ZVAR	0.999 (1.07)	1.783 (1.17)	1.826 (1.15)	0.495 (1.20)	1.336 (1.21)	1.519 (1.38)
GDP Growth						
OLS	2.819 (1.32)	3.122 (1.52)	2.982 (1.39)	0.954 (1.35)	2.156 (1.48)	2.509 (1.49)
DOLS-AIC	2.266 (1.06)	2.140 (1.04)	2.322 (1.08)	0.696 (0.99)	1.465 (1.00)	1.641 (0.98)
Litterman	2.340 (1.09)	2.332 (1.13)	2.459 (1.15)	0.800 (1.13)	1.776 (1.22)	1.878 (1.12)
Modified Litterman	2.237 (1.05)	2.036 (0.99)	2.133 (0.99)	0.710 (1.00)	1.430 (0.98)	1.621 (0.96)
ZVAR	2.141	2.058	2.147	0.706	1.456	1.681
Partial ZVAR	2.250 (1.05)	2.351 (1.14)	2.447 (1.14)	0.785 (1.11)	1.775 (1.22)	1.919 (1.14)

<sup>a</sup> The numbers in parentheses give the ratio of the RMSE of the associated model to the RMSE of the ZVAR forecasts at each horizon. A value greater than one means that the RMSE of the given model is larger than for the ZVAR forecasts, indicating that the given model's forecast is less accurate than the ZVAR forecasts.

Sources: Unemployment and CPI, Bureau of Labor Statistics; GDP, Bureau of Economic Analysis

dominated by all the others. The results are described in more detail below.<sup>10</sup>

*Unemployment.* The modified Litterman model generates the most accurate unemployment forecasts of any of the alternative forecast schemes considered here. However, the ZVAR model performs quite well, having an RMSE no more than 5 percent higher than the modified Litterman model. The Litterman and partial ZVAR models' forecasts are essentially equivalent, but each performed worse than either the ZVAR or modified Litterman models. For example, for the one-year annual forecast, the RMSE from the Litterman specification is 14 percent higher than from the ZVAR model, and the RMSE from the partial ZVAR model is 10 percent higher. The DOLS-AIC is the next best performing specification, but it has an RMSE more than 30 percent higher than the ZVAR for one- and two-year annual forecasts. The OLS model performs very poorly for any variable over any horizon other than the current quarter. For each of the one- and two-year annual forecasts the RMSE is over 40 percent higher than for the ZVAR model.

*CPI Inflation.* The ZVAR and modified Litterman specifications generate almost equally accurate infla-

tion forecasts at all horizons, with the relative RMSEs differing by no more than 3 percent. The DOLS-AIC specification is marginally more accurate than either the ZVAR or modified Litterman model at the two-quarter and the one-year horizons. But the improvement in RMSE is no more than 3 percent over the ZVAR benchmark. The partial ZVAR generates somewhat worse forecasts than the Litterman model, especially for annual forecasts. More notably, the RMSE for the partial ZVAR is almost 40 percent higher than for the ZVAR model for two-year annual forecasts. The OLS model generates by far the worst-performing inflation forecasts, with an RMSE 132 percent higher than the ZVAR model for two-year annual forecasts.

*GDP Growth.* The ZVAR and modified Litterman specifications generate almost equally accurate forecasts of GDP growth, with the RMSEs differing by no more than 5 percent. The DOLS-AIC specification also yields almost the same forecast accuracy for the annual forecasts, and it has an RMSE that is no more than 8 percent higher than the ZVAR model for the quarterly forecasts. The Litterman and partial ZVAR models are the next most accurate, but each have RMSEs that are

as much as 20 percent higher than the ZVAR model for annual forecasts. Again the OLS model performs the worst, having an RMSE for the two-year annual GDP growth forecast that is almost 50 percent higher than that from the ZVAR model.

The results suggest that using the sum of coefficients restriction, either exactly as in the case of the DOLS-AIC model or slightly more loosely as in the ZVAR and modified Litterman models, can significantly improve forecast performance over specifications that do not use such information. However, the fact that the restriction is not imposed exactly and that the cointegration prior is used might account for why the ZVAR and modified Litterman models decisively outperform the DOLS-AIC model in forecasting the unemployment rate.<sup>11</sup>

## Conclusion

This article illustrates in some detail the steps involved in one approach to producing real-time forecasts from a VAR model. It focuses attention on the technical hurdles that must be addressed in a real-time application and methods for overcoming those hurdles. The solutions to technical difficulties include conditional forecasting to handle the staggered release of data and the interpolation of lower-frequency data to match the frequency of monthly data. In addition, the article discusses methods that attempt to improve VAR forecast accuracy by imposing inexact prior restrictions.

The goal is to provide a road map for an analyst interested in designing and building a VAR forecasting model using these techniques.

The article then provides some suggestive empirical evidence regarding the performance of various possible specifications of a six-variable VAR in forecasting unemployment, inflation, and output growth. The forecast accuracy results show that using a particular setting for the systemwide inexact prior restrictions of the type described in Sims and Zha (1998) generates more accurate forecasts for unemployment, inflation, and real GDP growth than a VAR that uses the single-equation Litterman (1980) inexact priors. However, this improvement is largely explained by the incorporation of reasonably tight priors on the long-run properties of the VAR. This long-run aspect of the specification appears to matter more for the improvements in accuracy than the systemwide nature of the formulation does. In particular, a modified Litterman model that also incorporates these priors appears to be at least as accurate as the ZVAR model.

VAR models are increasingly being used for forecasting in private business and in policy institutions. It is hoped that the empirical techniques presented in this article will prove useful to those interested in implementing or at least understanding real-time forecasting with a VAR model.

10. The impact of conducting a real-time forecasting experiment is most noticeable in the short-term GDP forecasts. In particular, the quarterly GDP forecasts can be more accurate one or two quarters ahead than they are for the current quarter. However, when the same experiment is conducted using the July 1998 vintage of historical data throughout, the forecast accuracy uniformly declines as one forecasts beyond the current quarter. Recall that a GDP estimate is revised on at least three occasions, and the size of the revisions to the monthly series used in constructing the monthly GDP data is often quite large. Of the series used to distribute GDP data over a quarter, industrial production in particular is often substantially revised (see Robertson and Tallman 1998 for discussion of this point). These real-time data revision issues affect the near-term (current and next-quarter) forecasting accuracy statistics in a way not captured in a forecasting analysis that uses only the latest available data.

11. This article has not searched across parameter settings for the non-OLS VAR specifications that might have improved their performance further.

# Data Definitions

## Variables Included in the VAR

**Real GDP:** The value in real (1992) dollars of the output produced over a given quarter reported at a seasonally adjusted annual rate. Real GDP is measured in chain-weighted dollars to account for changes in relative prices over time. Availability is discussed below. Source: Bureau of Economic Analysis.

**Civilian unemployment rate:** The percentage of the civilian labor force that is unemployed. Seasonally adjusted. Released on either the first or second Friday of the month following the month measured. Source: Bureau of Labor Statistics.

**Price level:** Consumer price index (CPI) for all urban consumers. Not seasonally adjusted. Available by about the middle of the month following the measured month. Currently the average CPI value for the years from 1982 to 1984 is set equal to 100. A not seasonally adjusted series is used in the empirical analysis described in the article largely because it was readily available over the forecast period 1986 to 1997. However, there appears to be little seasonality in U.S. CPI data. Source: Bureau of Labor Statistics.

**M2 money stock:** Seasonally adjusted. Measured in billions of current dollars and available around the middle of the month after the month to which they refer. The aggregate is currently composed of the sum total of coins and paper currency, traveler's checks, demand deposits, other checkable deposits (NOW, share drafts), overnight repurchase agreements, overnight Eurodollars, general purpose and broker/dealer money market funds, money market deposit accounts, savings deposits, and small-denomination time deposits. Source: Board of Governors of the Federal Reserve System.

**Effective federal funds rate:** The interest charged between banks on loans of reserves held with the Federal Reserve System and measured as the monthly average of federal funds transactions for a group of federal funds brokers who report to the Federal Reserve Bank of New York each day. Source: Board of Governors of the Federal Reserve System.

**Commodity prices (CP):** Spot raw industrial subindex of thirteen markets for commodity prices. Not seasonally adjusted. Compiled daily and available at a monthly frequency at the end of the current month. The thirteen included prices are burlap, scrap copper, cotton, hides, lead scrap, print cloth, rosin, rubber, steel scrap, tallow, tin, wool tops, and zinc. Source: Commodity Research Bureau.

## Monthly Series Used in the Chow-Lin Distribution Procedure

**Nonagricultural payroll employment:** Available on the first or second Friday following the measured month. Measured in millions of employed. Seasonally adjusted. Source: Bureau of Labor Statistics.

**Total industrial production index:** Available midmonth following the month measured. Measured as an index of physical output produced in a selection of sectors. 1987 = 100. Seasonally adjusted. Source: Board of Governors of the Federal Reserve System.

**Real personal consumption expenditures:** Seasonally adjusted. New estimates are usually available by the end of the month after that being measured. Source: Bureau of Economic Analysis.

## Data Release Sequence

There is usually a delay of a few weeks between the end of a quarter and the release of the initial estimate of quarterly real GDP (the advanced estimate) for that quarter. Two revised real GDP estimates (preliminary and final) are released in the two subsequent months. Moreover, real GDP data for any particular quarter are subsequently revised, and this process of revision can continue many years after the initial data release (see Robertson and Tallman 1998 for a discussion).

Monthly civilian unemployment rate data are published on the first or second Friday of the month immediately following the month to which they refer, and the CPI data are published around the middle of the month following the month to which the data refer. There is generally little revision to the unemployment and CPI data over time, although major benchmark revisions to the CPI are made approximately every ten years.

The Board of Governors of the Federal Reserve System publishes monthly data on various money aggregates by the middle of the month after the month to which the data refer. These estimates are revised on a continuing basis with the receipt of more accurate source data, and on occasion the historical M2 data have been subject to major redefinition (see Anderson and Kavajecz 1994).

Numerous commodity price indexes are available on a daily basis. The study uses the monthly average of the daily closing CRB/BLS index of spot prices for raw industrial commodities as the commodity price. The federal funds rate data are the monthly average of the daily effective funds rate.

# The Chow and Lin (1971) Procedure for Distributing Quarterly GDP Observations

Chow and Lin (1971) derive a procedure for distributing quarterly observations on a flow variable across the months of a quarter. Their algorithm assumes that the  $T$  observations in the monthly series  $y_m$  to be estimated are related to  $T$  observations on  $n$  monthly indicator variables  $X_m$  via a regression of the form

$$y_m = X_m \beta + u_m, \tag{A1}$$

where  $y_m$  is  $T \times 1$ ,  $X_m$  is  $T \times n$ , and the regression error follows a stationary first-order autoregression  $u_{m,t} = \rho_m u_{m,t-1} + e_{m,t}$  for  $t = 1, \dots, T$ , with the  $T \times 1$  vector  $e_m$  having zero mean and a covariance matrix  $\sigma^2 I_T$ . Thus, the covariance matrix of  $u_m$  has the standard form  $V_m = [\sigma^2 / (1 - \rho_m^2)] P_m$ , where

$$P_m = \begin{bmatrix} 1 & \rho_m & \dots & \rho_m^{T-1} \\ \rho_m & 1 & & \rho_m^{T-2} \\ \vdots & & \ddots & \vdots \\ \rho_m^{T-1} & & & 1 \end{bmatrix}$$

(see for example, Hamilton 1994).<sup>1</sup>

The  $(T/3)$  quarterly observations are related to the monthly observations via an “averaging” matrix

$$C = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & \dots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & \dots & 1 & 1 & 1 \end{bmatrix}$$

This matrix implies a regression relationship for the quarterly observations of the form

$$y_q = C y_m = X_q \beta + u_q \tag{A2}$$

and where the covariance matrix of  $u^q$  will be

$$V_q = C V_m C'. \tag{A3}$$

Chow and Lin show that the smallest variance linear unbiased estimator of  $y_m$  is

$$\hat{y}_m = X_m \hat{\beta} + \hat{P}_m C' (C \hat{P}_m C')^{-1} \hat{u}_q,$$

where  $\hat{\beta}$  is the (generalized) least squares estimate of  $\beta$  in equation (A2). To estimate  $\rho_m$  notice that the auto-regression coefficient in the quarterly regression,  $\rho_q$ , is the ratio of the first to the second elements of the first row of  $V_q$  in equation (A3). This ratio reveals that  $\rho_m$  can be obtained as the unique solution to the polynomial

$$\rho_q = \frac{\rho_m^5 + 2\rho_m^4 + 3\rho_m^3 + 2\rho_m^2 + \rho_m}{2\rho_m^2 + 4\rho_m + 3}$$

and replacing  $\rho_q$  with the (generalized) least squares estimate of  $\rho_q$  in equation (A2) provides a consistent estimate of  $\rho_m$ .

1. If  $X_m$  contains a stochastic trend, then the regression represents a cointegrating relationship. A modification of the Chow and Lin procedure that was suggested by Litterman (1983) would be appropriate in the case that  $\rho_m = 1$ .

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