MINIMAL STATE VARIABLE SOLUTIONS TO MARKOV-SWITCHING RATIONAL EXPECTATIONS MODELS

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Abstract. We develop a new method for deriving minimal state variable (MSV) equilibria of a general class of Markov switching rational expectations models and a new algorithm for computing these equilibria. We compare our approach to previously known algorithms, and we demonstrate that ours is both efficient and more reliable than previous methods in the sense that it is able to find MSV equilibria that previously known algorithms cannot. Further, our algorithm can find all possible MSV equilibria in models. This feature is essential if one is interested in using a likelihood based approach to estimation.

I. Introduction

For at least twenty five years, economists have estimated structural models with constant parameters using U.S. and international data. Experience has taught us that some parameters in these models are unstable and a natural explanation for the failure of the parameter constancy assumption is that the world is changing. There are competing explanations for the source of parameter change that include abrupt breaks in the variance of structural shocks (Stock and Watson, 2003; Sims and Zha, 2006; Justiniano and Primiceri, 2008), breaks in the parameters of the private sector equations due to financial innovation (Bernanke, Gertler, and Gilchrist, 1999; Christiano, Motto, and Rostagno, 2008; Gertler and Kiyotaki, 2010), or breaks in the parameters of monetary and fiscal policy rules (Clarida, Galí, and Gertler, 2000; Lubik and Schorfheide, 2004; Davig and Leeper, 2007; Fernandez-Villaverde and Rubio-Ramirez, 2008; Christiano, Eichenbaum, and Rebelo, 2009). Markov-switching rational expectations (MSRE) models can capture the fact that the structure of the economy changes over time.

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Cogley and Sargent (2005a)'s estimates of random coefficient models suggest that when parameters change, they move around in a low dimensional subspace; that is, although all of the parameters of a VAR may change – they change together. This is precisely what one would expect if parameter change were due to movements in a small subset of parameters of a structural rational expectations model. Although this phenomenon can be effectively modeled as a discrete Markov process, Sims (1982) and Cooley, LeRoy, and Raymon (1984) pointed out some time ago that a rational expectations model should take account of the fact that agents will act differently if they are aware of the possibility of regime change.

In a related paper (Farmer, Waggoner, and Zha, 2009), we show that equilibria of MSRE models are of two types; minimal state variable (MSV) equilibria and non-fundamental equilibria. Non-fundamental equilibria may or may not exist. If a non-fundamental equilibrium exists, it is the sum of an MSV equilibrium and a secondary stochastic process. Our innovation in this paper is to develop an efficient method for finding MSV equilibria in a general class of MSRE models, including those with lagged state variables. Given the set of MSV equilibria, our earlier paper (Farmer, Waggoner, and Zha, 2009) shows how to construct non-fundamental equilibria.

Previous authors, notably Leeper and Zha (2003), Svensson and Williams (2005), Davig and Leeper (2007), and Farmer, Waggoner, and Zha (2008) have made some progress in developing methods to solve for the equilibria of MSRE models. But the techniques developed to date are not capable of finding all of the equilibria in a general class of MSRE models. We illustrate this point with an example. We use a simple rational expectations model to illustrate why previous approaches (including our own) may not find an MSV equilibrium, and in the case of multiple MSV equilibria, can at best find only one MSV equilibrium. In contrast, we show that our new method is able to find all MSV equilibria. The algorithm we develop is shown to be fast and efficient.

II. Minimal state variable solutions

A general class of MSRE models studied in the literature has the following form:

$$\overset{A(s_t)}{\begin{bmatrix} \underset{(n-\ell)\times n}{a_1(s_t)} \\ \underset{\ell\times n}{a_2(s_t)} \end{bmatrix}} \underset{n\times 1}{x_t} = \overset{B(s_t)}{\begin{bmatrix} \underset{(n-\ell)\times n}{b_1(s_t)} \\ \underset{\ell\times n}{b_2(s_t)} \end{bmatrix}} \underset{n\times 1}{x_{t-1}} + \overset{\Psi(s_t)}{\begin{bmatrix} \underset{(n-\ell)\times k}{\psi_1(s_t)} \\ \underset{\ell\times k}{\psi_2(s_t)} \end{bmatrix}} \underset{k\times 1}{\varepsilon_t} + \overset{\Pi(s_t)}{\begin{bmatrix} \underset{(n-\ell)\times \ell}{\pi_1(s_t)} \\ \underset{\ell\times \ell}{\pi_2(s_t)} \end{bmatrix}} \underset{\ell\times 1}{\eta_t}, \tag{1}$$

where x_t is an $n\times1$ vector of endogenous and predetermined variables, $a_1, a_2, b_1, b_2, \psi_1, \psi_2, \pi_1$, and π_2 are conformable parameter matrices, ε_t is a $k \times 1$ vector of i.i.d. stationary exogenous shocks, and η_t is an $\ell \times 1$ vector of expectational errors. The variable s_t is an exogenous stochastic process following an h-regime Markov chain, where $s_t \in \{1, ...h\}$ with transition matrix $P = [p_{ij}]$ defined as

$$p_{ij} = \Pr(s_t = i \mid s_{t-1} = j).$$

Because the vector η_t is a mean zero endogenous stochastic process and we implicitly assume that Π_{s_t} is of full rank, without loss of generality we let $\pi_1(s_t) = 0$, $\pi_2(s_t) = I_\ell$, $\psi_1(s_t) = \psi(s_t)$, and $\psi_2(s_t) = 0$, where I_ℓ is the $\ell \times \ell$ identity matrix.

In most applications, x_t is partitioned as

$$x_t' = \begin{bmatrix} y_t' & z_t' & E_t y_{t+1}' \end{bmatrix}, \tag{2}$$

where the first pair $[y_t' \quad z_t']$ is of dimension $n-\ell$ and the second block of Equation (1) is of the form $y_t = E_{t-1}y_t + \eta_t$. The vector y_t is the endogenous component and z_t is the predetermined component consisting of lagged and exogenous variables. In this case, the endogenous shocks η_t can be interpreted as expectational errors. Regime-switching constant terms can be encoded by introducing a dummy variable $z_{c,t}$ as an element of the vector z_t together with the additional equation $z_{c,t} = z_{c,t-1}$, subject to the initial condition $z_{c,0} = 1$. While this addition introduces a unit eigenvalue into the system, the solution techniques developed in this paper are not affected because the dummy variable is just a constant term and the stationarity of the system is intact.

In Farmer, Waggoner, and Zha (2009), we develop a set of necessary and sufficient conditions for equilibria to be determinate in a class of forward-looking MSRE models. We show in that paper that every solution of an MSRE model, including an indeterminate equilibrium, can be written as the sum of an MSV solution and a secondary stochastic process (i.e., the sunspot component). For models with lagged state variables, the most challenging task is to find all MSV equilibria; this task has not been successfully accomplished in the literature. Once an MSV equilibrium is found, the secondary stochastic process is straightforward to obtain, as shown in Farmer, Waggoner, and Zha (2009).

To give a precise description of an MSV equilibrium in an MSRE model, we first consider the constant parameter case, a special case of the Markov-switching system given by (1), which we represent as follows,

$$\overset{A}{\begin{bmatrix} \underset{(n-\ell)\times n}{a_1} \\ \underset{\ell\times n}{a_2} \end{bmatrix}} \underset{n\times 1}{x_t} = \overset{B}{\begin{bmatrix} \underset{(n-\ell)\times n}{b_1} \\ \underset{\ell\times n}{b_2} \end{bmatrix}} \underset{n\times 1}{x_{t-1}} + \overset{\Psi}{\begin{bmatrix} \underset{(n-\ell)\times k}{\psi} \\ \underset{\ell\times k}{0} \end{bmatrix}} \underset{k\times 1}{\varepsilon_t} + \overset{\Pi}{\begin{bmatrix} \underset{(n-\ell)\times \ell}{0} \\ I_\ell \end{bmatrix}} \underset{\ell\times 1}{\eta_t}. \tag{3}$$

There are a variety of techniques to solve this system and the general solution is of the form

$$x_t = \Gamma x_{t-1} + \Xi_1 \varepsilon_t + \Xi_2 \gamma_t, \tag{4}$$

where the mean-zero random process γ_t, if present, is a sunspot component. For expositional clarity, let us assume that A is invertible. The matrices Γ, Ξ_1, and Ξ_2 can be obtained from the real Schur decomposition of $A^{-1}B = UTU'$. The matrix U is orthogonal and T is block upper triangular with 1×1 and 2×2 blocks along its diagonal. The 1×1 blocks correspond to real eigenvalues of $A^{-1}B$ and the 2×2 blocks correspond to conjugate pairs of complex eigenvalues of $A^{-1}B$. The real Schur decomposition is unique up to the ordering of the eigenvalues along the block diagonal of T. If we partition U as $U = [V \quad \hat{V}]$, then the Schur decomposition can be written as

$$A^{-1}B = \begin{bmatrix} V & \hat{V} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} V' \\ \hat{V}' \end{bmatrix}.$$

If we define $\Gamma = VT_{11}V'$, $\Xi_1 = VG_1$, and $\Xi_2 = VN_1$, where G_1 and N_1 are solutions of the matrix equations

$$\begin{bmatrix} AV & \Pi \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \Psi \text{ and } \begin{bmatrix} AV & \Pi \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = 0,$$

then Equation (4) will define a solution of the system given by (3). This is straight forward to verify by multiplying Equation (4) by A and then transforming the right hand side using the definitions of Γ, Ξ_1, and Ξ_2, the fact that x_t is in the column space of V, the identity $A^{-1}BV = \Gamma V$ and the implicit definition $\eta_t = -G_2\varepsilon_t - N_2\gamma_t$. Furthermore, any solution will correspond to some ordering of the eigenvalues $A^{-1}B$ and a partition of U. Since we require solutions to be stable,[1] all the eigenvalues of T_{11} must lie inside the unit circle.

The first requirement of an MSV solution is that it be fundamental, i.e. it cannot contain a sunspot component. This implies that N_1 must be zero or equivalently that $[AV \quad \Pi]$ must be of full column rank. The second requirement is that if x_t is decomposed as an endogenous component, a predetermined component, and an

[1]For constant parameter systems such (3), stable and bounded are equivalent requirements, but not so for the time varying systems such as (1).

expectations component as in Equation (2), then no restrictions should be placed on the "data", which corresponds to the endogenous and predetermined components. This implies that the number of columns in V must be $n-\ell$ and that $[AV \quad \Pi]$ be invertible.

We can use these ideas to formalize what we mean by an MSV equilibrium. First, note that the column space of V is the span of solution x_t in the sense that support of the random process x_t is contained in and spans the column space of V. A solution of the system (3) is an MSV solution if and only if it is the unique solution on its span and there are no restrictions on the endogenous and predetermined variables y_t and z_t. On the span $E_t y_{t+1}$ is a function of y_t and z_t. These ideas can be expanded to the Markov switching system given by (1) and (2). In this context, the relevant concept is not the span of the solution, but the conditional span. The span of the solution x_t conditional on $s_t = i$ is the span of the support of the random process x_t given $s_t = i$.

Definition 1. *A stable solution of the system given by* (1) *and* (2) *is a minimal state variable solution if and only if it is unique given all the conditional spans and none of the conditional spans impose a relationship among the endogenous and predetermined components y_t and z_t.*

Unlike the constant parameter case, one can no longer apply an eigenvalue condition used to identify all candidates for the conditional spans. One can, however, use iterative techniques to construct MSV equilibria. Our approach builds on the following theorem.

Theorem 1. *If $\{x_t, \eta_t\}_{t=1}^{\infty}$ is an MSV solution of the system* (1)*, then*

$$x_t = V_{s_t} F_{1,s_t} x_{t-1} + V_{s_t} G_{1,s_t} \varepsilon_t, \tag{5}$$

$$\eta_t = -\left(F_{2,s_t} x_{t-1} + G_{2,s_t} \varepsilon_t\right), \tag{6}$$

where the matrix $\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix}$ is invertible and

$$\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix} \begin{bmatrix} F_{1,i} \\ F_{2,i} \end{bmatrix} = B(i), \tag{7}$$

$$\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix} \begin{bmatrix} G_{1,i} \\ G_{2,i} \end{bmatrix} = \Psi(i), \tag{8}$$

$$\left(\sum_{i=1}^{h} p_{i,j} F_{2,i}\right) V_j = 0_{\ell,n-\ell}, \text{ for } 1 \le j \le h. \tag{9}$$

The dimension of V_i is $n \times (n-\ell)$, $F_{1,i}$ is $(n-\ell) \times n$, $F_{2,i}$ is $\ell \times n$, $G_{1,i}$ is $(n-\ell) \times k$, and $G_{2,i}$ is $\ell \times k$.

To find an MSV equilibrium, we must find matrices V_i such that $[A(i)V_i \quad \Pi]$ is invertible and Equation (9) holds where $F_{2,i}$ is defined via Equation (7). Since $\Pi = [0_{\ell,n-\ell} \;\; I_\ell]'$, the matrix $[A(i)V_i \quad \Pi]$ is invertible if and only if the upper $(n-\ell)\times(n-\ell)$ block of $A(i)V_i$ is invertible. It is easy to see that multiplying V_i on the right by an invertible matrix, and hence multiplying $F_{1,i}$ and $G_{1,i}$ on the left by the inverse of this matrix, will not change equations (5) through (9). Thus, without loss of generality, we assume that

$$A(i)V_i = \begin{bmatrix} I_{n-\ell} \\ -X_i \end{bmatrix} \tag{10}$$

for some $\ell \times (n-\ell)$ matrix X_i. Since

$$\begin{aligned} F_{2,i} &= \begin{bmatrix} 0_{\ell,n-\ell} & I_\ell \end{bmatrix} \begin{bmatrix} A(i)V_i & \Pi \end{bmatrix}^{-1} B(i) \\ &= \begin{bmatrix} X_i & I_\ell \end{bmatrix} B(i), \end{aligned}$$

Equation (9) becomes

$$\sum_{i=1}^{h} p_{ij} \begin{bmatrix} X_i & I_\ell \end{bmatrix} B(i)A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix} = 0_{\ell,n-\ell}. \tag{11}$$

In the previous derivation, we assume that $A(i)$ is invertible for expositional clarity. In Appendix B, we remove this assumption and show that our iterative algorithm works even if $A(i)$ is not invertible.

The advantage of our method is that we are able to reduce the task of finding an MSV solution to that of computing the roots of a quadratic polynomial in several variables. We exploit Newton's method to compute these roots. This has the advantage over previously suggested methods of being fast and locally stable around any given solution. This property guarantees that by choosing a large enough grid of initial conditions we will find all possible MSV solutions. This local convergence property does not hold for iterative solutions that have previously been suggested in the literature.

Let $X = (X_1, \cdots, X_h)$, define f_j to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{\ell(n-\ell)}$ given by

$$f_j(X) = \sum_{i=1}^{h} p_{ij} \begin{bmatrix} X_i & I_\ell \end{bmatrix} B(i)A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix}, \tag{12}$$

and f to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{h\ell(n-\ell)}$ given by

$$f(X) = (f_1(X), \cdots, f_h(X)). \tag{13}$$

The quadratic polynomial equations, $f(X) = 0$, are the same as the constraints represented by (9).

Thus, finding an MSV equilibrium is equivalent to finding the roots of $f(X)$ and Theorem 1 suggests the following constructive algorithm for finding MSV solutions.

Algorithm 1. *Let $X^{(1)} = \left(X_1^{(1)}, \cdots, X_h^{(1)}\right)$ be an initial guess. If the k^{th} iteration is $X^{(k)} = \left(X_1^{(k)}, \cdots, X_h^{(k)}\right)$, then the $(k+1)^{th}$ iteration is given by*

$$vec\left(X^{(k+1)}\right) = vec\left(X^{(k)}\right) - f'\left(X^{(k)}\right)^{-1} vec\left(f\left(X^{(k)}\right)\right).$$

where

$$f'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial X_1}(X) & \cdots & \frac{\partial f_1}{\partial X_h}(X) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_h}{\partial X_1}(X) & \cdots & \frac{\partial f_h}{\partial X_h}(X) \end{bmatrix}.$$

The sequence $X^{(k)}$ converges to a root of $f(X)$.

It is straightforward to verify that for $i \neq j$,

$$\frac{\partial f_j}{\partial X_i}(X) = p_{ij}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(i)A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix}\right)' \otimes I_\ell$$

and for $i = j$,

$$\frac{\partial f_j}{\partial X_j}(X) = p_{jj}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(j)A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix}\right)' \otimes I_\ell + I_{n-\ell} \otimes \left(\sum_{k=1}^{h} p_{kj} \begin{bmatrix} X_k & I_\ell \end{bmatrix} B(k)A(j)^{-1} \begin{bmatrix} 0_{n-\ell,\ell} \\ -I_\ell \end{bmatrix}\right).$$

In a series of computational experiments, reported below, we have found that this algorithm is relatively fast and that it converges to multiple solutions, when they exist, for a suitable choice of initial conditions.

Once an MSV equilibrium is obtained, one can verify whether this solution is stationary (mean-square-stable) in the sense of Costa, Fragoso, and Marques (2004, page 36). Let $\Gamma_j = V_j A(j)$ for $j = 1, \ldots, h$. As shown in Costa, Fragoso, and Marques (2004, Proposition 3.9, p. 36 and Proposition 3.33, p.49), an MSV solution is stationary if and only if the eigenvalues of

$$(P \otimes I_{n^2})\, diag\left[\Gamma_1 \otimes \Gamma_1, \ldots, \Gamma_h \otimes \Gamma_h\right], \tag{14}$$

are all inside the unit circle.

In Section IV, we present simple examples in which existing algorithms, that have been proposed in the literature, break down. We also show that when there are multiple MSV equilibria, existing algorithms can at best find only one equilibrium and sometimes do not converge to any MSV equilibrium even when the initial starting point is close to the equilibrium. This result is unsatisfactory because researchers should be able to estimate models by searching across the space of all equilibria and selecting the one that maximizes the posterior odds ratios. In all the examples we study, our algorithm is capable of finding all MSV equilibria by randomly choosing different initial points.

III. Previous approaches

Two existing algorithms have been frequently used to find an MSV equilibrium in a MSRE model: the fixed-point (FP) algorithm developed in a previous version of this paper (Farmer, Waggoner, and Zha (2008)) and the iterative algorithm proposed by Svensson and Williams (2005). We review these algorithms in this section and in Section IVwe discuss why they do not always work well in practice.

III.1. **The FP algorithm.** To apply the FP algorithm, Farmer, Waggoner, and Zha (2008) show how to define an expanded state vector $\tilde{x}_t$. Using their definition, one can write the Markov switching equations as a constant parameter system of the form

$$\tilde{A}\tilde{x}_t = \tilde{B}\tilde{x}_{t-1} + \tilde{\Psi}\tilde{u}_t + \tilde{\Pi}\eta_t, \tag{15}$$

where $\tilde{x}_t \in R^{nh}$ has dimension $nh \times 1$.

To write system 1 in this form, define a family of matrices $\{\phi_i\}_{i=1}^{h}$ where h is the number of Markov states and each ϕ_i has dimension $\ell \times n$ with full row rank. Define $\mathbf{e}_j$ as a column vector equal to 1 in the j^{th} element and zero everywhere else and the matrix Φ as

$$\underset{\ell(h-1)\times nh}{\Phi} = \begin{bmatrix} \mathbf{e}_2' \otimes \phi_2 \\ \vdots \\ \mathbf{e}_h' \otimes \phi_h \end{bmatrix}. \tag{16}$$

Let the matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{\Pi}$ be given by

$$\underset{nh\times nh}{\tilde{A}} = \begin{bmatrix} diag\left(a_1(1), \cdots, a_1(h)\right) \\ a_2 \quad \cdots \quad a_2 \\ \Phi \end{bmatrix},$$

$$\underset{nh\times nh}{\tilde{B}} = \begin{bmatrix} diag\left(b_1(1), \cdots, b_1(h)\right)\left(P \otimes I_n\right) \\ b_2 \quad \cdots \quad b_2 \\ 0 \end{bmatrix},$$

$$\underset{nh\times\ell}{\tilde{\Pi}} = \left[\begin{array}{c} 0, I_\ell, 0 \end{array}\right]'.$$

To define $\tilde{u}_t$ and the corresponding coefficient matrix $\tilde{\Psi}$, let $\mathbf{1}_h$ be the h-dimensional column vector of ones and let

$$\underset{(n-\ell)h\times nh}{S_i} = \left(diag\left[b_1\left(1\right),\cdots,b_1\left(h\right)\right]\right)\times\left[\left(\mathbf{e}_i\mathbf{1}_h' - P\right)\otimes I_n\right],$$

for $i = 1.\ldots,h$. With this notation, we have

$$\tilde{u}_t = \left[\begin{array}{c} S_{s_t}\left(\mathbf{e}_{s_{t-1}}\otimes\left(\mathbf{1}_h'\otimes I_n\right)\tilde{x}_{t-1}\right) \\ \mathbf{e}_{s_t}\otimes u_t \end{array}\right],$$

and

$$\underset{nh\times(k+n-\ell)h}{\tilde{\Psi}} = \left[\begin{array}{cc} I_{(n-\ell)h} & diag\left(\psi\left(1\right),\cdots,\psi\left(h\right)\right) \\ 0 & 0 \\ 0 & 0 \end{array}\right].$$

It is straightforward to show that $E_{t-1}\left[u_t\right] = 0$. Thus, (15) is a linear system of rational expectations equations and the solution of this linear system can be computed by known methods. Farmer, Waggoner, and Zha (2008), show that a solution of the expanded system (15) with the initial conditions x_0 and $\tilde{x}_0 = \mathbf{e}_{s_0}'\otimes x_0$ is a solution of the original nonlinear system. The vectors x_t and $\tilde{x}_t$ are related by the expression,

$$x_t = \left(\mathbf{e}_{s_t}'\otimes I_n\right)\tilde{x}_t. \tag{17}$$

Although (3) is a linear rational expectations system, finding $\{\phi_1,\phi_2,...\phi_h\}$ for this linear system is a fixed-point problem of a system of nonlinear equations. Farmer, Waggoner, and Zha (2008) propose the following algorithm. Let the superscript (n) denote the n^{th} step of an iterative procedure. Beginning with a set of initial matrices $\left\{\phi_i^{(0)}\right\}_{i=2}^{h}$, define $\Phi^{(0)}$ using Equation (16) and generate the associated matrix $A^{(0)}$. Next, compute the QZ decomposition of $\left\{A^{(0)},B\right\}$ and denote the generalized eigenvalues corresponding the unstable roots by $Z_u^{(0)} = \left[z_1^{(0)},\ldots,z_h^{(0)}\right]$, where $z_i^{(0)}$ is an $\ell\times n$ matrix. Finally, set $\phi_i^{(1)} = z_i^{(0)}$. Form this new set of values of ϕ_i's, form a new matrix $A^{(1)}$. Repeat this algorithm and, if it converges, the system (15) will generate sequences $\{x_t,\eta_t\}_{t=1}^{\infty}$ that are consistent with the system (1), where x_t is governed by (17).

The qualification *if it converges* is crucial because, as we will show in Section IV, it may not converge even in the simplest rational expectations model.

III.2. **The SW algorithm.** In this subsection we describe the algorithm developed by Svensson and Williams (2005). As we exhaust many commonly used mathematical symbols for matrices and vectors, we will use the same notation for some variables and parameters as in Section III.1 as long as this double use of the notation does not cause confusion.

Svensson and Williams (2005)'s algorithm is an iterative approach to solving a general Markov-switching system. The system is written as

$$X_t = A_{11,s_t} X_{t-1} + A_{12,s_t} x_{t-1} + C_{s_t} \epsilon_t, \tag{18}$$

$$E_t H_{s_{t+1}} x_{t+1} = A_{21,s_t} X_t + A_{22,s_t} x_t, \tag{19}$$

where X_t is an $n_X \times 1$ vector of predetermined variables, x_t is an $n_x \times 1$ vector of forward-looking variables, and s_t. The MSV solution takes the following form:

$$x_t = G_{s_t} X_t.$$

The algorithm works as follows.

(1) Start with an initial guess of $G_j^{(0)}$, where $s_t = j$.
(2) For $n = 0, 1, 2, \ldots$, iterate the value of $G_j^{(n+1)}$ according to

$$G_j^{(n+1)} = \left[A_{22,j} - \sum_k P_{kj} H_k G_k^{(n)} A_{12,k} \right]^{-1} \left[\sum_k P_{kj} H_k G_k^{(n)} A_{11,k} - A_{21,j} \right]. \tag{20}$$

This algorithm is both elegant and efficient and can handle a large system. If it converges to an MSV solution, the convergence is fast. As we show below, however, the algorithm may not converge even if there is an MSV equilibrium.

IV. Comparison of our algorithm with alternatives

In this section we illustrate the properties of different methods using three simple examples based on the following model:

$$\phi_{s_t} \pi_t = E_t \pi_{t+1} + \delta_{s_t} \pi_{t-1} + \beta_{s_t} r_t,$$

$$r_t = \rho_{s_t} r_{t-1} + \epsilon_t,$$

where $s_t = 1, 2$ takes one of two discrete values according to the Markov-switching process. If we interpret π_t as inflation and r_t as an exogenous shock to income or preferences, this equation can be derived directly from the consumer's optimization problem together with a monetary policy rule that moves the interest rate in response to current and past inflation rates (see Liu, Waggoner, and Zha (2009)).

IV.1. **An example with a unique MSV equilibrium.** We set $\delta_{s_t} = 0, \beta_{s_t} = \beta = 1$, and $\rho_{s_t} = \rho = 0.9$ for all values of s_t, $\phi_1 = 0.5, \phi_2 = 0.8, p_{11} = 0.8$, and $p_{22} = 0.9$. One can show that for this parameterization (i.e., $\delta_{s_t} = 0$), there is a unique MSV equilibrium.[2] The MSV solution has a closed form given by the expression,

$$\pi_t = g_{1,s_t} r_{t-1} + g_{2,s_t}\epsilon_t,$$

where

$$\begin{bmatrix} g_{1,1} \\ g_{1,2} \end{bmatrix} = \begin{bmatrix} p_{11}\rho - \phi_1 & p_{21}\rho \\ p_{12}\rho & p_{22}\rho - \phi_2 \end{bmatrix}^{-1} \begin{bmatrix} \beta\rho \\ \beta\rho \end{bmatrix},$$

$$g_{2,s_t} = \frac{p_{1s_t} g_{1,1} + P_{2s_t} g_{1,2} + \beta}{\phi_{s_t}}.$$

In experiments based on this example, our algorithm converged quickly to the following MSV equilibrium for all initial conditions,

$$\pi_t = -10.9285 r_{t-1} - 12.1428\epsilon_t, \text{ for } s_t = 1,$$
$$\pi_t = 8.3571 r_{t-1} + 9.2857\epsilon_t, \text{ for } s_t = 2.$$

Using (14), one can easily verify that this equilibrium is mean square stable.

Both the FP or the SW algorithms, however, are *unstable* when applied to this example. To gain an intuition of why these previous algorithms do not work, we map this example to the notation of the SW algorithm described in Section III.2:

$$H_k = 1, n_X = n_x = 1, X_t = r_t, x_t = \pi_t, A_{11,k} = \rho, A_{12,k} = 0, A_{21,j} = -\beta, A_{22,j} = \phi_j.$$

For expositional clarity, we further simplify the model by assuming that $\phi_1 = \phi_2 = \phi = 0.85$. The MSV equilibrium for this case can be characterized as

$$\pi_t = g_1 r_{t-1} + g_2\varepsilon_t,$$

where $g_1 = \frac{\beta\rho}{\phi-\rho}$. It follows from (20) that

$$g_1^{(n)} = \frac{\left(g_1^{(n-1)} + \beta\right)\rho}{\phi}.$$

The above iterative algorithm also characterizes the FP algorithm. Since the MSV solution g_1 is great than 1 in absolute value and $\rho/\phi > 1$ in this case, $g_1^{(n)}$ will go to either plus infinity or minus infinity (depending on the initial guess) as $n \to \infty$. Thus, the FP and SW algorithms cannot find the MSV equilibrium, even when there is only a unique MSV equilibrium.

[2]There also exists a continuum of non-fundamental equilibria around the unique MSV solution.

IV.2. **An example with two MSV equilibria.** We now provide an example where there are multiple MSV equilibria, but the SW algorithm can find only one of the two MSV equilibria and the FP algorithm cannot converge at all. In contrast, our proposed algorithm converges to all of the MSV equilibria by randomly selecting different sets of initial guesses. The example has the following parameter configuration:

$$\phi_1 = 0.5, \phi_2 = 0.8, \delta_1 = -0.7, \delta_2 = 0.4,$$
$$\beta_1 = \beta_2 = 1, \rho_1 = \rho_2 = 0, p_{11} = 1.0, p_{22} = 0.64.$$

One can easily verify that the first regime, taken in isolation, is determinate while the second regime is indeterminate. We choose this example to show that even though the first regime is an absorbing state because $p_{11} = 1.0$, the MSV equilibrium in the regime-switching environment is *not* unique. To see this point clearly, note that the MSV solution takes the form $\pi_t = g_{1,s_t}\pi_{t-1} + g_{2,s_t}\epsilon_t$ with two distinct stationary equilibria:

$$g_{1,1} = -0.623212, g_{1,2} = 0.675998, \text{ first MSV equilibrium;}$$
$$g_{1,1} = -0.623212, g_{1,2} = 0.924559, \text{ second MSV equilibrium;}$$

Note that the multiple equilibria occur only in the second regime. The equilibrium in the first regime is unique.

The SW algorithm cannot find the second equilibrium; it converges only to the first equilibrium. The FP algorithm fares worse. It cannot converge to either of the two MSV equilibria.

IV.3. **An example with more than two MSV equilibria.** We now provide an example that a multiplicity of MSV equilibria can exist. Both FP and SW algorithms can find only one of them. The question is whether our proposed algorithm is capable of finding all the solutions or only a subset of them.

The example has the following parameter configuration:

$$\phi_1 = 0.2, \phi_2 = 0.4, \delta_1 = -0.7, \delta_2 = -0.2,$$
$$\beta_1 = \beta_2 = 1, \rho_1 = \rho_2 = 0, p_{11} = 0.9, p_{22} = 0.8.$$

An MSV equilibrium takes the form $\pi_t = g_{1,s_t}\pi_{t-1} + g_{2,s_t}\epsilon_t$. For this example, there are four stationary MSV equilibria given by

$$g_{1,1} = -0.765149, g_{1,2} = -0.262196, \text{ first MSV equilibrium;}$$
$$g_{1,1} = 0.960307, g_{1,2} = 0.646576, \text{ second MSV equilibrium;}$$
$$g_{1,1} = -0.826316, g_{1,2} = 0.96551, \text{ third MSV equilibrium;}$$
$$g_{1,1} = 1.024809, g_{1,2} = -0.392746, \text{ fourth MSV equilibrium.}$$

Our algorithm converges rapidly to all the MSV solutions when we vary the initial guess randomly. In contrast, both the FP and SW algorithms, no matter what the initial guess (unless it is set exactly at an MSV solution), converge to only the first MSV equilibrium reported above.

Farmer, Waggoner, and Zha (2008) show an easy-to-check condition for the uniqueness of the equilibrium *if* it is found by the FP algorithm. This condition applies only to a local uniqueness and to the stacked linear system 15. This local results cannot be extended to the original Markov-switching system 1. Indeed, as this example shows, even the first MSV equilibrium is locally unique according to Farmer, Waggoner, and Zha (2008), there exist other MSV equilibria that are not in the neighborhood of the first equilibrium. Our new method is developed to find all possible MSV equilibria.

V. A general strategy of selecting an equilibrium

In this section we discuss a general strategy of selecting an equilibrium in the presence of multiple MSV equilibria. We first provide details of our efficient algorithm used for drawing initial guesses that cover a wide range of values in order to find all the MSV equilibria. After we have all the MSV equilibria in hand, we then propose a likelihood based criterion for selecting an MSV equilibrium while discussing other alternative criteria.

V.1. **Initial values.** Our new algorithm requires an initial guess in search of an equilibrium. A brute force approach is to simply use a large grid of initial values in a hope that different initial values may lead to different MSV equilibria. This approach is not a problem for a theoretical paper whose purpose is to highlight key properties of a particular model of interest. In an estimation exercise, however, this approach can become extremely inefficient when the size of a dynamic stochastic general equilibrium (DSGE) model is large.

Dan and Roger: please double check the proposed algorithm. An efficient approach is to randomly sample initial values by exploring the theoretical properties of the MSV solution. From the solution (5) one can see that V_i is uniquely determined only up to normalization discussed in Hamilton, Waggoner, and Zha (2007). Thus, we can always impose the restriction that the columns of V_i be orthonormal. Theorem 9 in Rubio-Ramírez, Waggoner, and Zha (2010) gives an efficient algorithm of implementing a random selection of V_i. Specifically, let $\tilde{X}_i$ be an $n \times n$ random matrix with each element having an independent standard normal distribution; and let $\tilde{X}_i = \tilde{Q}_i \tilde{R}_i$ be the QR decomposition of $\tilde{X}_i$ with the diagonal of $\tilde{R}_i$ normalized to be positive. Then the first $n - \ell$ columns of $\tilde{Q}_i$ form an independent random selection of V_i. The following algorithm gives a systematic way of finding all MSV equilibria.

Algorithm 2. *For each independent selection of V_i, we obtain the corresponding random selection of the initial value of X_i according to* (10).

(Step 1) Randomly draw $\tilde{N}$ initial values of $(X_1, \cdots, X_h)$.

(Step 2) For each initial value, apply apply Algorithm 1 to find an MSV equilibrium.

(Step 3) Collect all MSV equilibria.

(Step 4) Repeat Steps 1-3 with $\tilde{N} = 2 * \tilde{N}$ initial values.

(Step 5) Compare all MSV equilibria in Step 4 to the previously obtained MSV equilibria.

(Step 6) If they are the same, stop. If they are additional MSV equilibria found, go back to Steps 4 and 5.

Our experience indicates that with the starting number $\tilde{N} = 20$, it often takes no more than three repetitions for Algorithm 2 to converge.

V.2. **How to select a particular MSV equilibrium?** Once we obtain all MSV equilibria, a relevant question is: Which equilibrium should be selected? One answer is to follow the engineering literature (Costa, Fragoso, and Marques, 2004) and select the MSV equilibrium that is most stationary (i.e., the equilibrium with the smallest dominant eigenvalue (in absolute value) of the matrix (14)). The intuition is that this most stationary is likely to be most "attractive" in the sense that most initial guesses of X will converge to this equilibrium. It turns out that this intuition is not always true. To see this point, we conduct a heuristic exercise by randomly selecting 1000 initial values of X and tabulating the percentage in which a particular equilibrium the initial values converge to. For the example discussed in Section IV.2, the first equilibrium (with the dominant eigenvalue 0.388) receives 73% and the second equilibrium (with the dominant eigenvalue 0.547) receives 27%. For the example studied in Section IV.3, the first and second equilibria (with the dominant eigenvalues being 0.529 and 0.845 respectively) share the highest percentage of convergence and each receives 33%. The second highest percentage of convergence, 26%, goes to the third equilibrium (with the dominant eigenvalue 0.811). The fourth equilibrium (with the dominant eigenvalue 0.949) has the lowest percentage of convergence (8%). This example shows that a less stationary equilibrium can have the highest degree of attraction.

A better argument for selecting the most stationary MSV equilibrium is offered by Ellison and Pearlman (Forthcoming). They show that the most stationary MSV equilibrium is E-stable while other equilibria are not.[3] This is a persuasive argument from the view point of learning. For Markov-switching rational expectations models themselves, however, a more relevant question is based on the likelihood principle: Which equilibrium should be selected *conditional on the data we observe*? This alternative question is important because, ultimately, an equilibrium we select ought to explain the observed data.

We propose the following likelihood based approach. For each configuration of model parameters, we use Algorithms 1 and 2 to find all MSV equilibria. For each equilibrium, we compute the likelihood value recursively by following the method of Sims, Waggoner, and Zha (2008) (note that the prior density value is the same for all the equilibria). We compare all the likelihood values and select an equilibrium associated with the highest likelihood value. It is important to bear in mind that for a different configuration of model parameters due to parameter uncertainty, the nature of the selected equilibrium may be different as well.

VI. An application to a monetary policy model

In previous sections, we showed that the FP and SW algorithms may not converge to an MSV equilibrium and that if they converge, they converge to only one MSV equilibrium. In contrast, our new algorithm, using Newton's method to compute roots, is stable, efficient, and reliable for finding all MSV equilibria.

In this section we present simulation results based on a calibrated version of the New-Keynesian model and we use it to study changes in output, inflation, and the nominal interest rate.

Clarida, Galí, and Gertler (2000) and Lubik and Schorfheide (2004) argue that the large fluctuations in output, inflation, and interest rates are manifestations of indeterminacy induced by passive monetary policy. Sims and Zha (2006), on the

[3]Their theoretical results pertain only to a class of rational expectations models without Markov-switching parameters.

other hand, find no evidence in favor of indeterminacy when they allow monetary policy to switch regimes stochastically. Furthermore, they find that once the model permits time variation in disturbance variances, there is no evidence in favor of policy changes at all (see also Cogley and Sargent (2005b) and Primiceri (2005)).

Once it is known that policy changes might occur, a *rational* agent should treat these changes probabilistically and the probability of a future policy change should enter into his current decisions. Previous work in this area has neglected these effects and all of the studies cited above study regime switches in a purely reduced form model. We show in this section how to use the MSV solution to a MSRE model to study the effects of regime change that is rationally anticipated to occur. We use simulation results to show that the persistence and volatility in inflation and the interest rate can be the result of (1) policy changes, (2) changes in shock variances, or (3) changes in private sector parameters. Hence, our method provides a tool for empirical work, in which a more formal analysis of the data can be used to discriminate between these competing explanations.

Our regime-switching policy model, based on Lubik and Schorfheide (2004), has the following three structural equations:

$$x_t = E_t x_{t+1} - \tau(s_t)(R_t - E_t \pi_{t+1}) + z_{D,t}, \tag{21}$$

$$\pi_t = \beta(s_t) E_t \pi_{t+1} + \kappa(s_t) x_t + z_{S,t}, \tag{22}$$

$$R_t = \rho_R(s_t) R_{t-1} + (1 - \rho_R(s_t)) \left[\gamma_1(s_t)\pi_t + \gamma_2(s_t) x_t\right] + \epsilon_{R,t}, \tag{23}$$

where x_t is the output gap at time t, π_t is the inflation rate, and R_t is the nominal interest rate. Both π_t and R_t are measured in terms of deviations from the steady state.[4] The coefficient τ measures the intertemporal elasticity of substitution, β is the household's discount factor, and the parameter κ reflects the rigidity or stickiness of prices.

The shocks to the consumer and firm's sectors, $z_{D,t}$ and $z_{S,t}$, are assumed to evolve according to an AR(1) process:

$$\begin{bmatrix} z_{D,t} \\ z_{S,t} \end{bmatrix} = \begin{bmatrix} \rho_D(s_t) & 0 \\ 0 & \rho_S(s_t) \end{bmatrix} \begin{bmatrix} z_{D,t-1} \\ z_{S,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{D,t} \\ \epsilon_{S,t} \end{bmatrix},$$

where $\epsilon_{D,t}$ is the innovation to a demand shock, $\epsilon_{S,t}$ is an innovation to the supply shock, and $\epsilon_{R,t}$ is a disturbance to the policy rule. All these structural shocks are

[4]See Liu, Waggoner, and Zha (2009) for a proof that the steady state in this example does not depend on regimes.

i.i.d. and independent of one another. The standard deviations for these shocks are $\sigma_D(s_t)$, $\sigma_S(s_t)$, and $\sigma_R(s_t)$.

Lubik and Schorfheide (2004) estimate a constant-parameter version of this model for the two subsamples: 1960:I-1979:II and 1979:III-1997:IV. In our calibration we consider two regimes. The parameters in the first regime correspond to their estimates for the period 1960:I-1979:II and the parameters in the second regime correspond to those for 1979:III-1997:IV. The calibrated values are reported in Tables 1 and 2. The transition matrix is calculated by matching the average duration of the first regime to the length of the first subsample and by assuming that the second regime is absorbing to accommodate the belief that the pre-Volcker regime will never return:[5]

$$P = \begin{bmatrix} 0.9872 & 0 \\ 0.0128 & 1 \end{bmatrix}.$$

A simple calculation verifies that, if only one regime were allowed to exist (in the sense that a rational agent was certain that no other policy would ever be followed) the first regime would be indeterminate and the second would be determinate. When a rational agent forms expectations by taking account of regime changes, we need to know if there exist multiple MSV equilibria. In our computations we apply our method to this system with a large number of randomly selected starting points and we obtain multiple MSV solutions for some configurations of parameterization that we report below.

This kind of forward-looking model provides a natural laboratory to experiment with different scenarios in light of the debate on changes in policy or changes in shock variances. The estimates provided by Lubik and Schorfheide (2004) and reported in Tables 1 and 2 mix changes in coefficients related to monetary policy with changes in other parameters in the model, since Lubik and Schorfheide (2004) do not account for the effect of the probability of regime change on the current behavior. One variation in the structural parameter values is to let the coefficient on the inflation variable in the policy equation (23) change while holding all the other parameters fixed across the two regimes. Tables 3 and 4 report the parameter values corresponding to this scenario, in which all the other parameters take the average of the values in Tables 1 and 2 over the two regimes. We call this scenario "policy change only".

In a second scenario, "variance change only", we keep the value of the policy coefficient γ_1 at 2.19 for both regimes while letting the standard deviation σ_D in the first

[5]One could also match the average duration of the second regime to the length of the second subsample, which give $p_{22} = 0.9865$.

regime be five times larger than that in the second regime and keeping the value of σ_S at 0.3712 for both regimes.[6] The parameter values for this scenario are reported in Tables 5 and 6.

The last scenario we consider allows only the parameters in the private sector to change. We call it "private-sector change only". The idea is to study whether the persistence and volatility in inflation can be generated by changes in the private sector in a forward-looking model. We let the coefficient τ be 0.06137 in the first regime and 0.6137 in the second regime. Tables 7 and 8 report the values of all the parameters for this scenario. Similar results can be achieved if one lets the value of κ in the first regime be much smaller than that in the second regime.

Using the method discussed in Section II, we obtain two MSV equilibria that characterize the first two scenarios and a unique MSV equilibrium for the last two scenarios. Figures 1-3 display simulated paths of the output gap, the interest rate, and inflation under each of these scenarios. With the original estimates reported in Lubik and Schorfheide (2004), the largest eigenvalue for the matrix (14) is 0.8617 for one equilibrium and 0.7225 for the other. The dynamics are quite different for these two MSV equilibria. We display the simulated data based on the MSV equilibrium with the largest eigenvalue 0.8617. The top chart in Figure shows that the output gaps in the first regime display persistent and large fluctuations relative to their paths in the second regime. It is well known that the constant-parameter New-Keynesian model of this type is incapable of generating much of the difference in output volatility between the two regimes. This is certainly true for the equilibrium with the largest eigenvalue 0.7225. When taking regime switching into account, we have two MSV equilibria and the difference in output dynamics between two regimes shows up in one of the equilibria.

When we restrict changes to the policy coefficient γ_1 only, the results are very similar to the first scenario, implying it is the change in policy across regimes that causes macroeconomic dynamics to be different across regimes. For this policy-change-only scenario, we have two MSV equilibria, one with the largest eigenvalue of the matrix (14) being 0.8947 and the other equilibrium with 0.6972. The second chart from the top in Figure 1 report the dynamics of output in the MSV equilibrium with the largest eigenvalue 0.6972. As one can see, the volatility in output is similar across

[6]Sims and Zha 2006 find that differences in the shock standard deviation across regimes can be on the scale of as high as $10 - 12$ times. One could also decrease the difference in σ_D and increase the difference in σ_S or experiment with different combinations. Our result that changes in variances matter a great deal will hold.

the two regimes. In summary, the top two charts in Figure 1 demonstrate that one can obtain rich dynamics from different MSV equilibria. Thus, it is important that a method be capable of finding all MSV equilibria if one would like to confront the model with the data.

When we allow only variances to change (the third scenario), there is a unique MSV equilibrium. The solution to this model is obtained by using the standard solution method of Sims (2002) because $E_{t-1}\varepsilon_{i,t} = 0$ for $i \in \{R, D, S\}$ even though their variances switch regime and because the uniqueness of a solution depends only on the parameters that are time invariant. As one can see from the third chart in Figure 1, the volatility of output in the first regime is distinctly larger than that in the second regime. The difference in volatility of output across regimes disappears in the private-sector-change-only scenario (the fourth scenario), as shown in the bottom chart of Figure 1.

Figures 2-3 display the simulated dynamics of the interest rate and inflation for the four scenarios. In all scenarios, both inflation and the interest rate in the first regime display persistent and large fluctuations relative to their paths in the second regime. The degree of persistence and volatility in these variables in the first regime increases with persistence of the shock $z_{D,t}$ or $z_{S,t}$ and with the size of shock variance $\sigma_{D,t}$ or σ_{S_t}. Our final scenario is particularly interesting because, as illustrated by the bottom charts of Figures 2-3, even if there is no change in policy and in shock variances, inflation and the interest rate can have much larger fluctuations in the first regime than in the second regime when the parameters of the private sector equations are allowed to change across regimes.

These examples teach us that the sharply different dynamics in output, the interest rate, and inflation observed before and after 1980 could potentially be attributed to different sources. The methods we have developed here give researchers the tools to address this and other issues in a regime-switching rational expectations in which rational agents take into account the probability of regime change when forming their expectations.

VII. Conclusion

We have developed a new approach to solving a general class of MSRE models. The algorithm we have developed has proven efficient and reliable in comparison to the previous methods. We have shown that MSV equilibria can be characterized as a vector-autoregression with regime switching, of the kind studied by Hamilton (1989) and Sims and Zha (2006). Our new method provides tools necessary for researchers to solve and estimate a variety of regime-switching DSGE models.

Table 1. Model coefficients (original)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.69	0.77	0.997	0.77	0.17
Second regime	0.54	0.58	0.993	2.19	0.30

Table 2. Shock variances (original)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.68	0.82	0.60	0.27	0.87	0.23
Second regime	0.83	0.85	0.84	0.18	0.37	0.18

Table 3. Model coefficients (policy change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.6137	0.6750	0.9949	0.77	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

Table 4. Shock variances (policy change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.6206	0.205
Second regime	0.755	0.835	0.72	0.225	0.6206	0.205

TABLE 5. Model coefficients (variance change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.6137	0.6750	0.9949	2.19	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

TABLE 6. Shock variances (variance change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.3712	0.205
Second regime	0.755	0.835	0.72	1.125	0.3712	0.205

TABLE 7. Model coefficients (private sector change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.0614	0.6750	0.9949	2.19	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

TABLE 8. Shock variances (private sector change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.6206	0.205
Second regime	0.755	0.835	0.72	0.225	0.6206	0.205

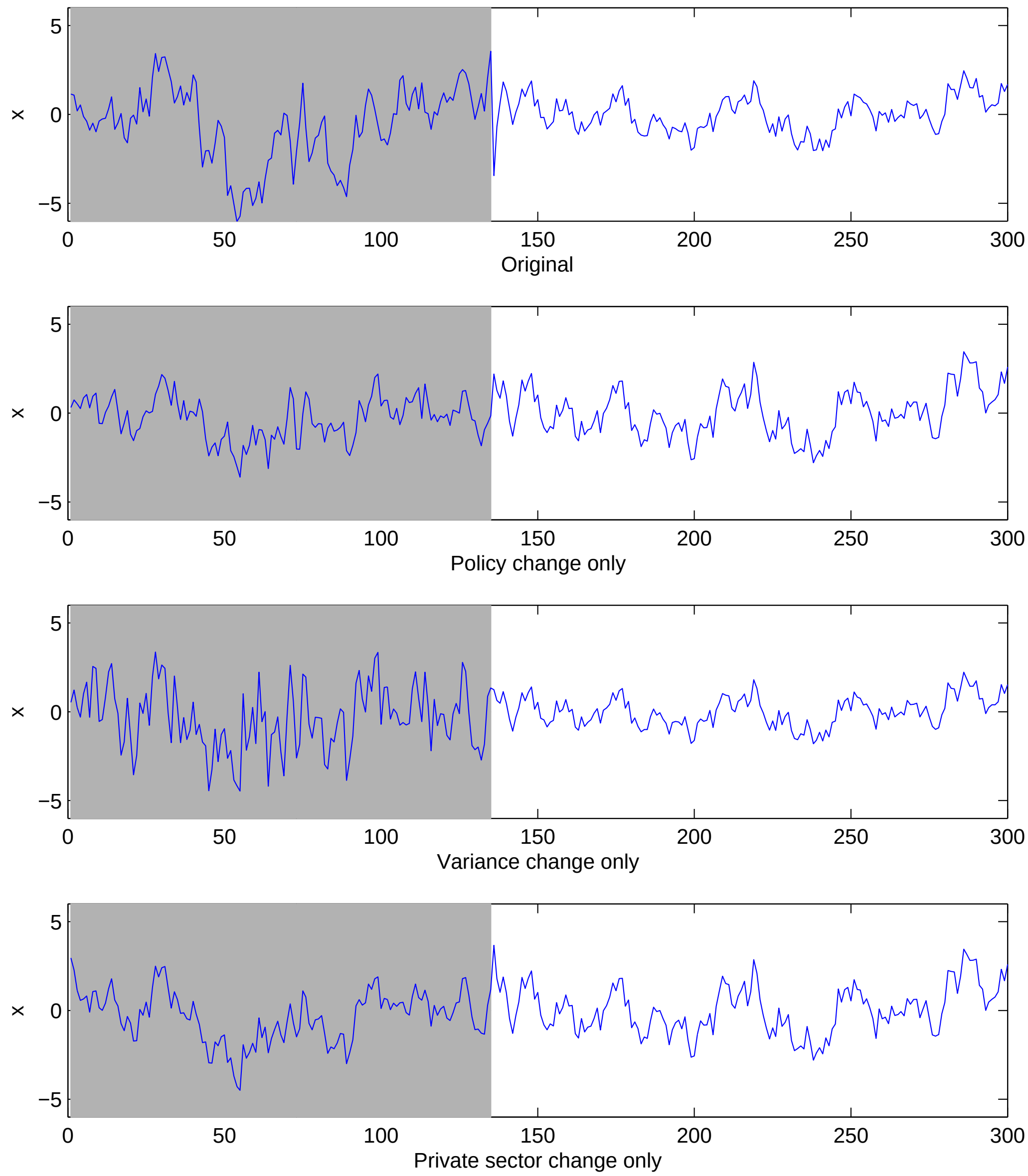

FIGURE 1. Simulated output gap paths from our regime-switching forward looking model. The shaded area represents the first regime.

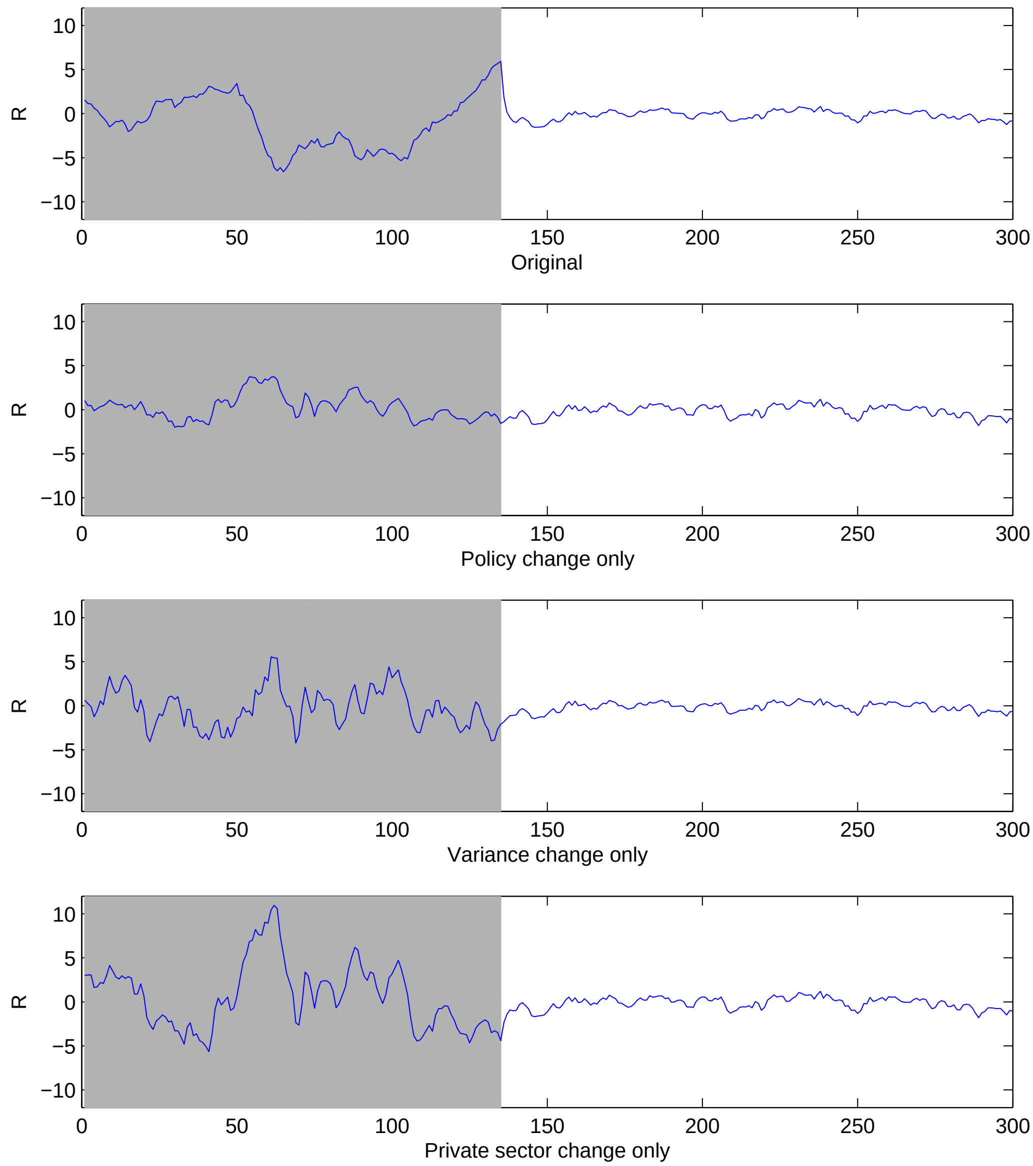

FIGURE 2. Simulated interest rate paths from our regime-switching forward looking model. The shaded area represents the first regime.

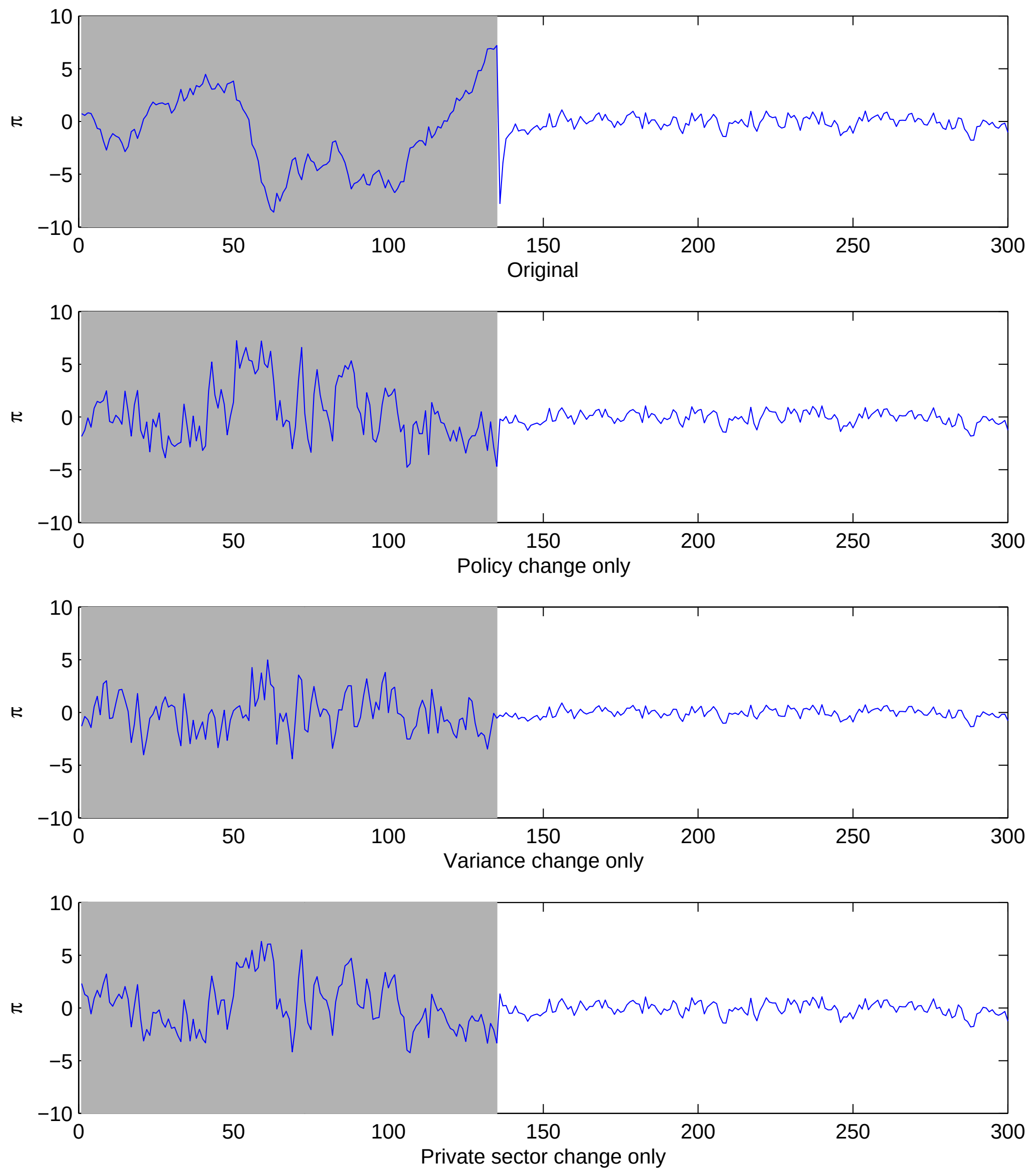

FIGURE 3. Simulated inflation paths from our regime-switching forward looking model. The shaded area represents the first regime.

Appendix A. Proof of Theorem 1

Let $\{x_t, \eta_t\}_{t=1}^{\infty}$ be an MSV solution of Equation (1). Denote the span of this solution, conditional on $s_t = i$, by $\hat{V}_i$ and let V_i be any $n \times (n-\ell)$ matrix whose columns form a basis for $\hat{V}_i$. Applying the $E_{t-1}[\cdot|s_t = i]$ operator to Equation (1) gives

$$A(i)E_{t-1}[x_t|s_t = i] = B(i)x_{t-1} + \Pi E_{t-1}[\eta_t|s_t = i]. \tag{A1}$$

This implies that for $1 \leq j \leq h$, every element of $B(i)V_j$ is a linear combination of the columns of the matrix $\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix}$. Thus there exist $(n-\ell) \times (n-\ell)$ matrices $F_{1,i,j}$ and $\ell \times (n-\ell)$ matrices $F_{2,i,j}$ such that

$$\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix} \begin{bmatrix} F_{1,i,j} \\ F_{2,i,j} \end{bmatrix} = B(i)V_j. \tag{A2}$$

Furthermore, since

$$\begin{aligned} \sum_{i=1}^{h} p_{i,s_{t-1}} A(i)E_{t-1}[x_t|s_t = i] &= \sum_{i=1}^{h} p_{i,s_{t-1}} \left(B(i)x_{t-1} + \Pi E_{t-1}[\eta_t|s_t = i] \right) \\ &= \sum_{i=1}^{h} p_{i,s_{t-1}} B(i)x_{t-1} + \Pi E_{t-1}[\eta_t] \\ &= \sum_{i=1}^{h} p_{i,s_{t-1}} B(i)x_{t-1} \end{aligned}$$

and Π is of full column rank, we can choose the $F_{1,i,j}$ and $F_{2,i,j}$ so that

$$\sum_{i=1}^{h} p_{i,j} F_{2,i,j} = 0_{\ell,n-\ell}.$$

Subtracting Equation (A1) from Equation (1) gives

$$A(i)\left(x_t - E_{t-1}[x_t|s_t = i]\right) = \Psi(i)\varepsilon_t + \Pi\left(\eta_t - E_{t-1}[\eta_t|s_t = i]\right).$$

This implies that there exist $(n-\ell) \times k$ matrices $G_{1,i}$ and $\ell \times k$ matrices $G_{2,i}$ such that

$$\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix} \begin{bmatrix} G_{1,i} \\ G_{2,i} \end{bmatrix} = \Psi(i). \tag{A3}$$

Let V_i^* denote the generalized inverse of V_i and define

$$\begin{aligned} \hat{x}_t &= V_{s_t} F_{1,s_t,s_{t-1}} V_{s_{t-1}}^* \hat{x}_{t-1} + V_{s_t} G_{1,s_t} \varepsilon_{t-1}, \\ \hat{\eta}_t &= -\left(F_{2,s_t,s_{t-1}} V_{s_{t-1}}^* \hat{x}_{t-1} + G_{2,s_t} \varepsilon_{t-1} \right). \end{aligned}$$

This will also be a solution of Equation (1) whose span, conditional on $s_t = i$, is $\hat{V}_i$. This can be verified by direct substitution using Equations (A2) and (A3) and the fact that $V_{s_{t-1}} V^*_{s_{t-1}} \hat{x}_{t-1} = \hat{x}_{t-1}$. Since $\{x_t, \eta_t\}_{t=1}^{\infty}$ is an MSV solution, it must be the case that $\hat{x}_t = x_t$ and $\hat{\eta}_t = \eta_t$.

Finally, $\begin{bmatrix} A(i)V_i & \Pi \end{bmatrix}$ must be invertible because otherwise we would have multiple solutions with the same conditional span. So, define

$$\begin{bmatrix} F_{1,i} \\ F_{2,i} \end{bmatrix} = \begin{bmatrix} A(i)V_i & \Pi \end{bmatrix}^{-1} B(i).$$

It is easy to see that $F_{1,i}V_j = F_{1,i,j}$ and $F_{2,i}V_j = F_{2,i,j}$. Thus

$$\left(\sum_{i=1}^{h} p_{i,j} F_{2,i} \right) V_j = 0_{\ell, n-\ell},$$

and

$$\begin{aligned} x_t &= V_{s_t} F_{1,s_t} x_{t-1} + V_{s_t} G_{1,s_t} \varepsilon_{t-1}, \\ \eta_t &= -\left(F_{2,s_t} x_{t-1} + G_{2,s_t} \varepsilon_{t-1} \right). \end{aligned}$$

Appendix B. Singular $A(i)$

Using the notation of Section II, we know that

$$A(i)V_i = \begin{bmatrix} I_{n-\ell} \\ -X_i \end{bmatrix}. \tag{A4}$$

If $A(i)$ were non-singular, then Equation (A4) is easily solved and the results of Section II follow. We now consider the case in which $A(i)$ may be singular. We can use the QR decomposition to find an invertible matrix U_i such that $A(i)U_i$ is of the form

$$\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \\ C_{1,i} & C_{2,i} \end{bmatrix}.$$

If the QR decomposition of $A(i)'$ is

$$A(i)' = Q_i R_i = Q_i \begin{bmatrix} R_{i,1} & R_{i,2} \\ 0_{\ell,n-\ell} & R_{i,3} \end{bmatrix},$$

then

$$U_i = Q_i \begin{bmatrix} \left(R'_{i,1}\right)^{-1} & 0_{n-\ell,\ell} \\ 0_{\ell,n-\ell} & I_\ell \end{bmatrix},$$

is the required matrix. If $R_{i,1}$ were not invertible, then $a_1(i)$, the upper block of $A(i)$, would not be of full row rank. This would imply an accounting identity exists, at least for this regime, among the endogenous and predetermined components. If this identity held across all regimes, which is the likely case, then the number of endogenous and predetermined variables could be reduced and the technique could proceed. Equation (A4) implies that

$$U_i^{-1}V_i = \begin{bmatrix} I_{n-\ell} \\ -Z_i \end{bmatrix}$$

for some $\ell \times n-\ell$ matrix Z_i and that $X_i = C_{i,2}Z_i - C_{i,1}$. Substituting this into Equation (9), we obtain

$$\sum_{i=1}^{h} p_{ij} \begin{bmatrix} C_{i,2}Z_i - C_{i,1} & I_\ell \end{bmatrix} B(i)U_j \begin{bmatrix} I_{n-\ell} \\ -Z_j \end{bmatrix} = 0_{\ell,n-\ell}.$$

Let $Z = (Z_1, \cdots, Z_h)$, define g_j to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{\ell(n-\ell)}$ given by

$$g_j(Z) = \sum_{i=1}^{h} p_{ij} \begin{bmatrix} C_{i,2}Z_i - C_{i,1} & I_\ell \end{bmatrix} B(i)U_j \begin{bmatrix} I_{n-\ell} \\ -Z_j \end{bmatrix} = 0_{\ell,n-\ell},$$

and g to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{h\ell(n-\ell)}$ given by

$$g(Z) = (g_1(Z), \cdots, g_h(Z)).$$

We now have the following algorithm for finding MSV solutions.

Algorithm 3. *Let $Z^{(1)} = \left(Z_1^{(1)}, \cdots, Z_h^{(1)}\right)$ be an initial guess. If the k^{th} iteration is $Z^{(k)} = \left(Z_1^{(k)}, \cdots, Z_h^{(k)}\right)$, then the $(k+1)^{th}$ iteration is given by*

$$vec\left(Z^{(k+1)}\right) = vec\left(Z^{(k)}\right) - g'\left(Z^{(k)}\right)^{-1} vec\left(g\left(Z^{(k)}\right)\right).$$

where

$$g'(X) = \begin{bmatrix} \frac{\partial g_1}{\partial Z_1}(Z) & \cdots & \frac{\partial g_1}{\partial Z_h}(Z) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_h}{\partial Z_1}(Z) & \cdots & \frac{\partial g_h}{\partial Z_h}(Z) \end{bmatrix}.$$

The sequence $Z^{(k)}$ converges to a root of $g(Z)$.

As before, it is straightforward to verify that for $i \neq j$,

$$\frac{\partial g_j}{\partial Z_i}(Z) = p_{ij}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(i)U_j \begin{bmatrix} I_{n-\ell} \\ -Z_j \end{bmatrix}\right)' \otimes C_{i,1}$$

and for $i = j$,

$$\frac{\partial g_j}{\partial Z_j}(Z) = p_{jj}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(j)U_j \begin{bmatrix} I_{n-\ell} \\ -Z_j \end{bmatrix}\right)' \otimes C_{j,1}$$
$$+ I_{n-\ell} \otimes \left(\sum_{k=1}^{h} p_{kj} \begin{bmatrix} C_{k,1}Z_k + C_{k_2} & I_\ell \end{bmatrix} B(k)U_j \begin{bmatrix} 0_{n-\ell,\ell} \\ -I_\ell \end{bmatrix}\right).$$

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